DRAFT - 12 Sept 2008
Susanna S. Epp

What Is a Variable?

Many years ago, when I was just beginning to think seriously about issues involved in teaching mathematical reasoning, I happened to read an article by the mathematician Paul Halmos in which he compared the role variables play in mathematics to the role pronouns play in ordinary language. As an example, Halmos (1977) gave the sentence

(1a) He who hesitates is lost,

which he rephrased as

(1b) For all x, if x hesitates then x is lost.

The philosopher W. O. Quine (1988) also discussed the relation between variables and pronouns, writing that variables are used to make cross-references in mathematics in much the same way that pronouns are used for cross-referencing in ordinary language. For example, consider the sentence

(2a) Are there one or more numbers with the following property: two times it plus 3 gives the same result as squaring it?

In this sentence the reference to "a number" that is achieved using "it" can be expressed even more clearly using a variable:

(2b) Are there one or more numbers x with the property that $2x + 3 = x^2$?

By serving as temporary names for such unknown quantities, variables enable us to work with them concretely, thereby helping us to discover what their value or values might be. The mathematician Jean Dieudonné (1972) pointed out the boldness of this usage: When we solve an equation, we operate with "the unknown (or unknowns) *as if it were a known quantity*. . .A modern mathematician is so used to this kind of reasoning that his boldness is now barely perceptible to him."

Observe, however, that the ability of pronouns to express cross-references unambiguously is limited. For instance, consider the statement

> No matter what nonzero real number is chosen, some real number exists such that its product with it equals 1.

To what do the words "it" and "its" refer – the original number or the one whose existence is claimed? Using variables makes this clear:

> No matter what nonzero real number r is chosen, some real number s exists so that $rs = 1$.

Both pronouns, such as "he," "she," "it," and "they," and variables, such as x, r, k, b_3, can be used as temporary names to refer back to many different objects in many different circumstances. But the large variety of symbols available for use as variables make it much easier for variables to keep cross-references straight than is possible with pronouns.

Variables as Placeholders

Linguists such as Steven Pinker have pointed out the difference between a "referential pronoun" and a word such as the "He" in "He who hesitates is lost." A referential pronoun refers to a uniquely identified individual or group of individuals. For example, in the sentence "Tom took off his coat as he entered the room" both "his" and "he" refer to the single individual Tom.

Similarly in the sentence "Tom took off his coat, hat, and gloves because they made him too hot" the word "they" refers to the specific collection consisting of Tom's coat, hat, and gloves.

By contrast, Pinker (1994) asked us to consider the example

Everyone returned to their seats.

Focusing on the meaning of the word "their," he observed that the sentence means the same as, "For all *X*, *X* returned to *X*'s seat," and he went on to write: "The "*X*" does not refer to any particular person or group of people; it is simply a placeholder…the *X* that comes back to a seat is the same *X* that owns the seat that *X* comes back to."[1]

A look back at the way variables are used in the previous examples shows that all of them can, in fact, be interpreted as placeholders. For instance, in the sentence "For all *x*, if *x* hesitates then *x* is lost," the letter *x holds a place* into which we may substitute the name of any person. In the sentence "Are there one or more numbers *x* with the property that $2x + 3 = x^2$?" our challenge is to discover which number or numbers we could put *in place of x* to obtain a true equation. The statement "No matter what nonzero real number *r* is chosen, some real number *s* exists so that $rs = 1$" asserts that no matter what real number we put *in place of r*, there is a real number that can be put *in place of s* so that the equation $rs = 1$ will be true. In each case the variables hold a place for the possible values of the things being discussed.

Some authors have suggested that because the word variable suggests variation, which may not be appropriate in all the situations in which variables are used, the term "literal" or "literal symbol" should be used instead. But, for better or worse, the word variable has become standard and the terms literal and literal symbol are essentially absent from ordinary mathematics instruction, although it is possible to avoid using any of these terms in the early grades.

The Different Uses of Variables

Schoenfeld and Arcavi (1988) reported on the considerable variety of responses they received from a diverse group of people – mathematicians, mathematics educators, computer scientists, linguists, and logicians – when asked to use a single word to describe the meaning of variable. Usiskin (1988) described the variety of situations in which variables are used and stated that inadequate attention to familiarizing students with their differences can cause problems. A thesis of this article is that thinking of variables as placeholders actually illuminates their use in the many different situations in which they arise and that it is often the language used to describe the situations that contributes most significantly to the problems students experience.

Variables Used to Express Unknown Quantities

When we say, "Solve $\sqrt{4-3x} = x$," what we mean is "Find all numbers (if any) that can be substituted in place of *x* in the equation $\sqrt{4-3x} = x$ in a way that makes its left-hand side equal to its right-hand side." The role of *x* as a placeholder in a situation like this is sometimes highlighted by replacing *x* by an empty box: $\sqrt{4-3\cdot\square} = \square$. The emptiness of the box is intended to invite the reader to imagine filling it in with a variety of different values, some of which would make the two sides equal and others of which would not.

[1] Pinker was making the point that because the word "their" in this sentence is not a referential pronoun, it does not have to agree in number with "Everyone."

A criticism one might make about elementary algebra courses is that while they generally do a good job in explaining the meaning of "Solve for *x*" at an early point in the course, they move quickly to using this terminology exclusively. In doing so, they ignore the fact that students absorb the new phrases at different rates and, as a result, may not have developed a deep understanding for the meaning of "Solve for *x*" during the initial period of instruction. Because subsequent problems stress mechanical procedures rather than their underlying significance, some students come to view *x* as a mysterious object with no relation to the world as they know it and the process of equation solving as a purely formal exercise done to satisfy a teacher rather than to answer meaningful questions. A remedy is for both textbooks and teachers in the classroom to reinforce the meaning of "Solve for *x*" by continuing occasional use of other more descriptive phrases, such as "Which number or numbers can be substituted in place of *x* to make the left-hand side of the equation equal to its right-hand side?"

Variables Used to Express Universal Statements

Most mathematical definitions, axioms, and theorems are examples of universal statements. For example, the distributive property for real numbers states that for all real numbers a, b, and c, $(a + b)c = ac + bc$. This means that no matter what real numbers are substituted in place of a, b, and c, the two sides of the equation $(a + b)c = ac + bc$ will be equal, assuming that the same numbers are substituted for both occurrences of a, b, and c.

Representing variables using empty boxes may also assist students' appreciation for the role they play in expressing universal statements. For example, the distributive property can be stated as follows:

$\square\cdot\Diamond + \square\cdot\Delta = \square\cdot(\Diamond + \Delta)$ no matter what real number we place in boxes $\square$, $\Diamond$, and Δ.

When such a formulation has been given, a teacher can use it to illustrate the large variety of different ways the distributive property can be applied by filling in the boxes with successively more complicated expressions:

$$2s + 2t = 2(s + t)$$

$$2x + 6 = 2x + 2\cdot 3 = 2(x + 3)$$

$$2^{100} + 2^{99} = 2\cdot 2^{99} + 1\cdot 2^{99} = (2 + 1)\cdot 2^{99} \quad [= 3\cdot 2^{99}]$$

$$(x^2 - 1)x + (x^2 - 1)(x - 3) = (x^2 - 1)(x + (x - 3)) \quad [= (x^2 - 1)(2x - 3)]$$

In mathematics classes it is common to abbreviate the statements of definitions and theorems, saying, for example, that a certain step of a solution is justified "because $(a + b)c = ac + bc$." Students are expected to understand without explicit instruction that this is a rephrasing – a kind of slangy version – of the distributive property. However, omitting the words "for all real numbers a, b, and c" can lead students to invest a, b, and c with meaning they do not actually have. The common description of a, b, and c as "general numbers" suggests that there is a category of number lying beyond the ordinary numbers with which we are familiar. For those whose sense of a, b, and c as placeholders is secure, this terminology will not be misleading, but those with a shakier sense of the meaning of variable may come to imagine a new and mysterious mathematical realm whose existence makes them uneasy.

The fact is that including quantification in mathematical statements and emphasizing the variables' role as placeholders removes much of their mystery. It is understandable that we as teachers want to avoid formality and not write more words than necessary. But as a result we have introduced conventions about how variables are to be quantified in various contexts, even though we do not state these conventions explicitly. As with all slang, some students pick up our meaning through examples, without overt instruction, but others are simply perplexed.

Variables Used as Generic Elements in Discussions

When we describe a variable as "generic," we mean that we are to think of it as a particular object that shares all the characteristics of every other object of its type but that has no additional properties. When variables are used in this way, they are sometimes called the "John Doe's of mathematics." For example, if we were asked to prove that the square of any odd integer is odd, we might start by saying, "Suppose n is any odd integer." As long as we deduce properties of n^2 without making any assumptions about n other than those satisfied by every odd integer, each statement we make about it will apply equally well to all odd integers. In other words, we could replace n by any odd integer whatsoever, and the entire sequence of deductions we might make about n would lead to a true conclusion.

As an example, consider that, by definition, for an integer to be odd means that it equals 2 times some integer plus 1. Because this definition applies to every odd integer, a proof might proceed as follows:

> *Proof*: Suppose n is any odd integer. By definition of odd, there is some integer m so that $n = 2m + 1$. It follows that
>
> $$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1.$$
>
> But $2m^2 + 2m$ is an integer, and so n^2 is also equal to 2 times some integer plus 1. Hence n^2 is odd.

Note that in this example, the numbers we would be allowed to substitute in place of m are entirely determined by the numbers we would have substituted in place of n. In fact $m = (n - 1)/2$.

Dieudonné's use of the word "boldness" to describe the process of solving an equation by operating with the variable as if it were a known quantity applies equally well to the use of variables as generic elements in proofs. For instance, by boldly giving the name n to an arbitrarily chosen but representative odd integer, we can investigate its properties it as if we knew what it was. And after we have used the definition of odd to deduce that n equals two times some integer plus 1, we can also boldly give that integer the name m so that we can work with it too as if we knew what it was.

Occasionally we may be given a problem in a way that asks us to think of a certain variable as generic right from the start. For instance, instead of being asked to prove that the square of any odd integer is odd, we might have given the problem: "Suppose n is any odd integer. Prove that n^2 is odd." In this case, we should already be thinking of n as capable of being replaced by any arbitrary odd integer, and we could respond by starting with the second sentence of the proof that is given above.

An important use of variables as generic elements occurs in deriving the equations of lines, circles, and other conic sections. For example, to derive the equation of the line through (3,1) with slope 2, we could proceed as follows: Suppose (x,y) is any point on the line. As long as we deduce properties of x and y without making any additional assumptions about their values, everything we conclude about (x,y) will be true no matter what point on the line might be substituted in its place.

We could continue by considering two cases: the first in which $(x,y) \neq (3,1)$ and the second in which $(x,y) = (3,1)$. For the first case, we note that what makes a straight line straight is the fact that its slope is the same no matter what two points are used to compute it. Therefore, if the slope is computed using (x,y) and (3,1), the result must equal 2:

$$\frac{y-1}{x-3}=2. \qquad \text{(Eqn. 1)}$$

Multiplying both sides by $x - 3$ gives

$$y - 1 = 2(x - 3). \qquad \text{(Eqn. 2)}$$

This concludes the discussion of the first case. In case $(x,y) = (3,1)$, both sides of equation (2) equal zero. Thus in the second case it is also true that $y - 1 = 2(x - 3)$. Therefore, because no assumptions about (x,y) were made except for its being a point on the line, we can conclude that every point (x,y) on the line satisfies the equation $y - 1 = 2(x - 3)$.

Note that we could also rewrite this example by using empty boxes to represent the variables. To do so we would start by writing: Suppose (□,◊) is any point on the line. Toward the end we would be able to say that every point (□,◊) on the line satisfies the equation ◊ – 1 = 2(□ – 3), and we could then rewrite this equation using traditional variable names in place of □ and ◊.

When students try to formalize universal statements, one problem they have is related to the use of variables as generic elements. For instance, students sometimes rewrite "All integers are rational" as "For all integers *x*, *x* are rational" – using the plural "are" rather than the singular "is." Actually, once they have written "For all integers *x*," they should think of *x* as singular but generic, capable of being replaced by any element in the set of integers.

Variables Used in Functional Relationships

As many authors have noted, understanding the use of variables in the definition of functions is critically important for students hoping to carry their study of mathematics to an advanced level. An equation such as $y = 2x + 1$ represents a functional relationship in the sense that for each possible change in the value of *x* there is a corresponding change in the value of *y*. However, to understand a sentence like this, students must understand that when we speak of "the value of *x*" or "the value of *y*" we mean the values that are put in their places.

It is especially in connection with functions that people describe a variable as "a quantity that can change" or say that the variables *x* and *y* "truly vary." But this terminology can lead students to think of variables as a bizarre new kind of being. The *x* or *y* (or whatever symbol is used) does not vary at all. It is only the quantities one is allowed to imagine replacing it that vary. The confusion engendered by this wording is similar to that exhibited by students who say that the number 0.99999… is "not equal to 1 but it gets closer and closer to 1" – as if "it" is moving along the number line. In a relationship like $y = 3x$, do we want students to realize that different replacements for *x* result in different replacement values for *y*? Absolutely! But do we want them to have the uneasy sense that *x* and *y* are themselves actually moving -- "wriggling on the page" as Ralph Raimi (1997/2001) scathingly put it? That can lead students to believe that variables exist in a mysterious universe beyond their understanding.

More than in other mathematical subjects, students must learn to translate the words we use when we describe a function into language that is meaningful to them. For example, we might refer to "the function $y = 2x + 1$." Taken by itself, however, "$y = 2x + 1$" is meaningless. Logicians refer to it as a predicate or open sentence. It only achieves meaning when particular numbers are substituted for the variables or when it is part of a longer sentence that includes words like "for all" or "there exists."

Students need to learn that when we write "the function $y = 2x + 1$," we mean "the relationship defined by corresponding to any given real number the real number obtained by multiplying the given number by 2 and adding 1 to the result." We think of x as holding the place for the number that we start with (and call it the "independent variable" because we can start with any real number) and y as holding the place for the number that we end up with (and call it the "dependent variable" because its value depends on the value of x). The specific letters used to hold the places for the variables have no meaning in themselves. For example, we could just as well write "the function "$v = 2u + 1$" or "the function "$q = 2p + 1$."

Another way to define this function is to call it "the function $f(x) = 2x + 1$"or, more precisely, "the function f defined by $f(x) = 2x + 1$ for all real numbers x." An advantage of this notation is that it leads us to think of the function as an object to which we are currently giving the name f. This notation also makes it natural for us to define "the value of the function f at x" as the number that is associated by the function f to the particular value that is put in place of x.

A variation of the preceding notation defines the function by writing $f(\square) = 2\square + 1$, pointing out that for any real number one might put into the box the value of the function is twice that number plus 1. The empty box representation is especially helpful for emphasizing the role of the variable as a placeholder in connection with composite functions. Students asked to find the rule for, say, $f(g(x))$ often become confused when both f and g have been defined by formulas that use x as the independent variable. But when the functions have been defined using empty boxes the relationships are clearer. Similarly, in a calculus class students find it easier to learn to compute, say $f(x + h)$, if they have previously been shown the definition of f using empty boxes.

Variables Used as Parameters

In the equation $y = kx$ we may consider k as a "parameter." This means that we are to imagine that we first substitute a number in place of k, and then, keeping that number fixed, we consider the equation

$$y = kx,$$

where we are allowed to substitute any numbers in place of x and y. However, the fact that a particular symbol is used as a parameter in one situation does not imply that it must be considered a parameter in another context. It should always be the obligation of a problem poser to indicate clearly which variables are to be regarded as parameters and which are not.

Dummy Variables and Questions of Scope

Dummy variables are just variables. We use the term "dummy" when we are especially concerned about problems that can result from thinking of them as more than just placeholders. For instance, we might state the definitions of even and odd integers as follows:

An integer is even if, and only if, there is an integer k so that the given integer equals $2k$.

An integer is odd if, and only if, there is an integer k so that the given integer equals $2k+1$.

Following such an introduction, many students try to prove that the sum of any even and any odd integer is odd by starting their argument as follows: "Suppose m is any even integer and n is any odd integer. Then $m = 2k$ and $n = 2k + 1$….."

Terms like "for all" and "for some" are called quantifiers. When we write statements starting "For all x" or "There exists x," the "scope of the quantifier" indicates how far into the statements the role played by the variable stays the same; we say that the variable is "bound" by the quantifier. In the case of the definitions of even and odd given above, the binding of each of the *k*'s extends only to the end of the sentence that contains them. It is essential for students to understand that the *k*'s are just placeholders, having no independent existence beyond these sentences. We can emphasize this fact by reformulating the definitions in a variety of different ways. For the definition of even, for instance, we could write:

For an integer to be even means that it equals twice some integer.

For an integer to be even means that there exists an integer that can be placed into box □ so that the given integer equals 2·□.

For an integer to be even means that it equals 2·□, for some placement of an integer into box □.

For an integer to be even means that it equals $2a$, where a is some integer.

For an integer to be even means that there exists an integer c so that the given integer equals $2c$.

For an integer n to be even means that there exists an integer s so that $n = 2s$.

Understanding variables as placeholders is equally important for interpreting summations and integrals. For example, given a sequence of real numbers $a_0, a_1, a_2, \ldots$, students need to understand that the following summations all represent the same quantity:

$$\sum_{k=1}^{10} a_k \,,\ \sum_{i=1}^{10} a_i \,.$$

Similarly, in a calculus class, they need to see that

$$\int_1^2 f(x)\,dx = \int_1^2 f(t)\,dt.$$

Variables Used in If-then Sentences

Consider the sentence: If x is prime, then $x + 1$ is even. A sentence like this can be thought of in two different ways, either as implicitly universally quantified (i.e., as a shorthand for "For all x, if x is prime then $x + 1$ is even") or as a "predicate" or "open sentence" (i.e., only meaningful if specific values are substituted for x). In the first case, the sentence is false because it alleges that "if x is prime, then $x + 1$ is even" is true for all substitutions of values for x, whereas when $x = 2$, x is prime and $x + 1$ is not even. In the second case, the sentence is neither true nor false: x is simply a placeholder into which various quantities may be substituted, and for certain substitutions, the resulting sentence is true whereas for others it is false.

Although sentences like "If x is prime, then $x + 1$ is even" are most often interpreted as universally quantified, it is important to give students some experience of them as open sentences in order for them to learn the logic that governs conditional statements.

Conclusion

A person with a deep understanding of concepts and relationships can tolerate many different abbreviated ways of expressing them. For instance, a mathematician is not bothered by the variety of different meanings of the symbol x in the sentence, "The slope of x^2 at $x = 3$ is

$\left[\dfrac{d(x^2)}{dx}\right]_3 = [2x]_3 = 6$." But a student may find such a sentence bewildering. As teachers, we should always try to look at what we present though the eyes of our students. While our aim is ultimately to lead them to understand mathematical terminology in all the ways it is used in practice, we also need to be sensitive to the extent to which the mathematical words we say might reasonably be interpreted in ways different from what we intend.

The following table summarizes some of the disparities between what we say and what we mean when we speak of variables.

What we say	**What we mean**
the value of x	the quantity that is put in place of x
as the value of x increases	as larger and larger numbers are put in place of x
as the value of x increases, the value of y increases	If larger and larger numbers are put in place of x, the corresponding numbers that are put in place of y become larger and larger
where x is any real number	for all possible substitutions of real numbers in place of x
Let n be any even integer.	Imagine substituting an integer in place of n but do not assume anything about its value except that it is an even integer.
By definition of even, $n = 2k$ for some integer k.	By definition of even, there is an integer we can substitute in place of k so that the equation $n = 2k$ will be true. (Note that there is only one such integer; its value is $n/2$.)
the function x^2	the function that relates each real number to the square of that number. In other words, for each possible substitution of a real number in place of x, the function corresponds the square of that number.
where x is some real number that satisfies the given property	There is a real number that will make the given property true if we substitute it in place of x.
A general linear function is a function of the form $f(x) = ax + b$ where a is any nonzero real number and b is any real number.	A general linear function is a function defined as follows: for all substitutions of real numbers in place of a and b (with $a \neq 0$), the function relates each real number to a times that number plus b. Or: the function is the set of ordered pairs where any real number can be substituted in place of the first element of the pair and the second element of the pair is a times the first number plus b.

This article has focused on the meaning of variables and the language used to describe them, but differences between what we say and how students interpret our words occur in many areas of mathematics. Sensitivity to such differences can be rewarding, both for enhancing students' learning and for increasing teaches' satisfaction.

REFERENCES

Dieudonne, J. (1972). Abstraction in mathematics and the evolution of algebra. In W. Lamon (Ed.), *Learning and the nature of mathematics* (pp. 102-103). Chicago: SRA Associates, Inc

Halmos, P. (1977). Logic from A to G. *Mathematics Magazine*, vol. 50: 5-11.

Pinker, S. (1994) *The Language Instinct: How the mind creates language* (pp. 370-403) New York, NY: Morrow.

Quine, W. O. (1982). *Methods of Logic*, 4th ed. (p. 133). Cambridge, MA: Harvard University Press.

Raimi, R. (1997, modified 2001). Notes from the NCTM volume, Research Issues in the Learning and Teaching of Algebra 1989, a symposium volume edited by Wagner and Kieran. http://www.math.rochester.edu/people/faculty/rarm/algebra.html (accessed 12 September 2008).

Schoenfeld, A. H. and Arcavi, A. (1988) On the Meaning of Variable. *The Mathematics Teacher*, vol. 81: 420-427.

Usiskin, Z. (1988). Conceptions of School Algebra and Uses of Variables. In Coxford, A. and Schulte, A. P. (Eds.), *Ideas of Algebra, K-12* (pp.8-19). Reston, VA: National Council of Teachers of Mathematics.