Two meanings of "proportional"[1]

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Abstract

In this paper we point out that there are two rather distinct mathematical meanings associated with statements of the form "*Y* is proportional to *X*". In clarifying these meanings we present an abstract, arithmetical treatment of proportionality in terms of sets of n-tuples of numbers. Given an n-tuple $A=(a_1,a_2,...,a_n)$, the "proportionality class" of *A* consists of the set *P(A)* of all n-tuples of the form $(sa_1,sa_2,...,sa_n)$, where *s* ranges over the positive real numbers. One meaning of "*Y* is proportional to *X*" is that *X* and *Y* are n-tuples that belong to the same proportionality class *P(A)*. Another meaning of "*Y* is proportional to *X*" is that *X* and *Y* are variable quantities sa_i and sa_j from the proportionality class *P(A).* An equivalent formulation of this second definition is that "*y* is proportional to *x*" when *y* and *x* are related by a homogeneous linear function of the form *y* = *kx*. In this case, the "constant of proportionality" of the relationship has the form $k = a_j / a_i$ when stated in terms of n-tuples of the class *P(A).*

1. Introduction

The notion of two quantities being "proportional" is widespread both in mathematics itself and in its applications. In this essay we point out that there are two rather distinct mathematical meanings associated with this term.

Here is one definition, adapted from the literature:

(1) Definition: Two n-tuples[2] $A=(a_1,a_2,...,a_n)$ and $B=(b_1,b_2,...,b_n)$ are ***proportional***[3] if there is a number *s* such that $(b_1,b_2,...,b_n)=(sa_1,sa_2,...,sa_n)$.

In such a case, the terms of one n-tuple are a *constant multiple* of the corresponding

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[1] Comments by Phil Daro, Vinci Daro, Emiliano Gomez, Jim Madden, and Mike Oehrtman on an earlier version of this paper have been very helpful.

[2] All numbers in this paper are assumed to be positive real numbers.

[3] In addition to "A and B are proportional" one also hears "A and B are ***proportional*** to each other" and "A is proportional to B" used with the same meaning. This variety of usages does not lead to difficulty here, since, as is easy to see, "proportional" in Definition 1 is a symmetric relation.

terms of the other. That is, $b_i = sa_i$ for $1 \le i \le n$. Equivalently, there is a *constant ratio* $b_i / a_i = s$ between corresponding terms of the two n-tuples.

Here are some examples that fit Definition (1).

(1a) In two similar triangles, corresponding sides are proportional.

To interpret such a statement in terms of the n-tuples of Definition (1), represent the sides of the two triangles with the triples (a_1, a_2, a_3) and (b_1, b_2, b_3). If s is a scale factor relating the sizes of these triangles, then (1a) says that $b_i = sa_i$ for $1 \le i \le 3$.

(1b) Each of the schools received funding proportional to its student population.

Let $(p_1, p_2, \ldots, p_n)$ represent the populations of the schools and $(m_1, m_2, \ldots, m_n)$ the money they receive. Then the statement says that $m_i = sp_i$ for $1 \le i \le n$, where s is the amount of money received per student by each school.

Here is another definition, also adapted from the literature:

(2) <u>Definition</u>: Two related variable quantities p and q are ***proportional***[4] if there is a positive constant k such that $q = kp$.

The term ***directly proportional*** is also used with the same meaning. The number k is called the *constant of proportionality* of the relationship. We take the view that k is a positive real number, and that the variable quantities p and q range over the positive real numbers.

Here are some examples that fit this definition.

(2a) In circles, the circumference[5] is proportional to the diameter.[6]

In this statement, circumference c and diameter d are understood as variable quantities linked by the formula $c = \pi d$, with constant of proportionality π.

(2b) In nuggets of pure gold, the mass is proportional to the volume.[7]

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[4] The remark of footnote 2 applies here as well.

[5] This statement uses the common and useful convention of letting *circumference* refer to *the length of the circumference* and *diameter* to *the length of the diameter*.

[6] One also hears: "In circles, circumferences are proportional to diameters", where the plurals indicate that it is many pairs (*circumference, diameter*) that are referred to.

[7] Here it is unusual to hear " In nuggets of pure gold, masses are proportional to volumes" with plurals. Compare (2a) and footnote 5. In fact, the use of singular terms to refer to variable quantities in this context is more widespread. Still, there is much variability in the use of the language of proportionality.

In this statement, mass m and volume v are understood as variable quantities linked by the formula $m = 19.8v$. The number 19.8 is the constant of proportionality of the relationship. Physically it is the *density* of gold in grams per cubic centimeter.

Definition (1) and Definition (2) each has at its core a formula of the general form $y = kx$. Still, an attempt to replace the two definitions with a single definition based on this formula runs into the problem that there is a fundamental difference in what the symbols x and y stand for in the two cases. In (1) they are "n-tuples" A and B, whereas in (2) they are "variable quantities" p and q. Further, in the n individual formulas of the form $b_i = sa_i$ that are implicit in Definition (1), the symbols b_i and a_i do not stand for "variable quantities".[8]

Instead of trying to explain away the differences, we take the position that Definitions (1) and (2) refer to two distinct situations where the language of proportionality is used. We will show that these two kinds of proportionality represent different aspects of a single underlying mathematical structure involving n-tuples.

Even though so far only Definition (1) refers directly to n-tuples, we will see that it is useful to re-formulate Definition (2) in terms of n-tuples as well. Briefly, while Definition (1) characterizes proportionality as a relationship between two *different* n-tuples, we will see that Definition (2) characterizes proportionality as a relationship among parts of the *same* n-tuple in a proportionality class of n-tuples.

In proceeding with a general treatment in terms of n-tuples, we will start by looking just at ordered pairs, or "2-tuples", a special case of an n-tuple. This is the subject of the next section.

2. *Proportionality in ordered pairs*

Here is Definition (1) stated for the particular case of ordered pairs: Two ordered pairs $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are ***proportional*** (to each other) if there is a positive real number s such that $(b_1, b_2) = (sa_1, sa_2)$. An alternative and useful way of formulating this definition is to start with a particular ordered pair (a_1, a_2), and then consider the set of *all* pairs (sa_1, sa_2) as s ranges over the positive real numbers:

Given a particular ordered pair (a_1, a_2), we refer to the set

(3) $\{(sa_1, sa_2) \mid \text{for } s > 0\}$

[8] In fact, the b_i and a_i are fixed quantities. It is the parameter s that varies (over the positive reals).

as the *proportionality class* of the ordered pair (a_1,a_2). This terminology is appropriate since, in the sense of Definition 1, every member of this set is proportional to every other member. In fact, it is easy to see that "proportional" is an *equivalence relation* on the set of all ordered pairs of positive real numbers, and that (3) is one of the equivalence classes of this relation.

Proportionality classes such as (3) are central to our treatment. Here are three different ways of looking at the set (3). It is the *equivalence class* of all ordered pairs proportional to (a_1,a_2); it is the set of all *dilated versions* of the ordered pair (a_1,a_2); and. it is the first quadrant portion of the *line* through (a_1,a_2) and the origin (0, 0).

The notion of a proportionality class as in (3) gives us a way of defining *proportional* for ordered pairs. This is a statement of Definition (1) for the particular case of ordered pairs.

(4) Definition: Two ordered pairs and (a_1,a_2) and (b_1,b_2) are ***proportional*** (to each other) if there is a positive real number *s* such that $(b_1,b_2)=(sa_1,sa_2)$. Equivalently, two ordered pairs (a_1,a_2) and (b_1,b_2) are ***proportional*** (to each other) if they belong to the same proportionality class of the form (3).

For example, the two ordered pairs (3, π) and (21, 7π) are proportional, since (21, 7π) = 7•(3, π). Each is a member of the proportionality class {(3s, πs) | for *s* > 0}.

Definition (4) gives the sense of *are proportional* that applies in a statement such as:

(5) In isosceles right triangles, corresponding sides are proportional.

To connect (5) to (4), represent two of the sides of isosceles right triangles as the ordered pairs (*leg, hypotenuse*). Since *hypotenuse* = $\sqrt{2}$ •*leg,* the proportionality class can be represented

(6) $\{(s,s\sqrt{2} \mid \text{ for } s>0\}$

For example, two particular members of the proportionality class are $(1,\sqrt{2})$ and $(7,7\sqrt{2})$. The two legs "correspond" 1↔ 7, as do their two hypotenuses $\sqrt{2}\leftrightarrow 7\sqrt{2}$. In our approach, the sense of *proportional* in a statement such as (5) corresponds to the fact that the 2-tuples $(1,\sqrt{2})$ and $(7,7\sqrt{2}))$ are *proportional* in the sense of Definition (4).

An important observation about a proportionality class of the form (3) is that, for all members of the class, there is a *constant ratio* (a_2/a_1) of the second term to the first. If we consider a *function* to be defined in terms of a set of ordered pairs, this observation gives us a way of looking at proportionality classes in terms of functions:

(7) The set $\{(sa_1, sa_2) \mid \text{for } s > 0\}$ from (3) is the set of ordered pairs defining the unique linear function $x \to {a_2}/{a_1}\, x$ from the positive reals to the positive reals that maps a_2 to a_1 and 0 to 0.

As *s* ranges over the positive reals, sa_1 and sa_2 are variable quantities that also range over the positive reals. With the interpretation (7) of a proportionality class as a function, we can now give a characterization of *proportional* that corresponds to that of Definition (2):

(8) <u>Definition</u>: Given the proportionality class $\{(sa_1, sa_2) \mid \text{for } s > 0\}$, we say that the variable quantity sa_2 is *proportional* to the variable quantity sa_1.

Using observation (7) we can now state

(9) The variable quantity *y* is *proportional* to the variable quantity *x* (in the sense of Definition (8)) if there is a positive constant *k* such that $y = kx$.

In other words, Definition (8) is equivalent to Definition (2), which is a characterization of *proportional* in wide use. The constant *k* of (9) is the *constant of proportionality* of the proportional relationship.

As an illustration, Definition (8) characterizes the sense of *is proportional to* that applies in a statement such as

(10) In isosceles right triangles, the hypotenuses are proportional to the corresponding legs.

Statement (10) refers to *all* isosceles right triangles, and "hypotenuses" and "legs" play the role of variable quantities. The "corresponding" leg for a hypotenuse is the leg in the same triangle. It is useful to look at this in terms of proportionality classes.

Statement (10) says that in the proportionality class (11) the second member, the variable quantity $\sqrt{2}\,l$ is proportional to the first member, the variable quantity *l*:

(11) $\{(l, \sqrt{2}l) \mid l > 0\}$

This relationship can be expressed in the function $h = \sqrt{2}l$. The quantities *h* and *l* are proportional in the sense that one is a constant multiple of the other. Proportionality in this case is a relationship between the two members of a pair in a set of ordered pairs. The number $\sqrt{2}$ in the function $h = \sqrt{2}l$ is the *constant of proportionality.*

Statement (10) and statement (5) are each about the hypotenuse and leg of isosceles right triangles, but it is important to see that they say very different things. To bring out the difference, let us interpret statement (5) in terms of the proportionality class (11).

Statement (5) says that, given any two members $(l_1, \sqrt{2}l_1)$ and $(l_2, \sqrt{2}l_2)$ of this proportionality class, we must have a positive real number *s* such that $(l_2, \sqrt{2}l_2) = s(l_1, \sqrt{2}l_1)$. In this case, $s = \frac{l_2}{l_1}$. In other words, $(l_1, \sqrt{2}l_1)$ and $(l_2, \sqrt{2}l_2))$ are proportional in the sense that one is a dilated version of the other. Proportionality in this case is a relationship between *different* ordered pairs. The number *s* serves as a *scale factor* relating the two pairs. It would never be called a constant of proportionality.

Perhaps the clearest way to emphasize the difference between statements (5) and (10) is in terms of similar figures in geometry: (5) refers to a relationship *between* two different figures, whereas (10) refers to a relationship *within* a single figure.

In summary, we have used the notion of a proportionality class of the form (3) to formulate two definitions of *proportional*, definitions (4) and (8).[9] It is clear that definitions (4) and (8) define different, but related things.

In order to further bring out both the differences and how they are related, in (12) we state them side by side, and also use language, in the underlined terms below, that explicitly differentiates the type of proportionality in each.

(12) <u>Definition</u>: In the proportionality class $\{(s, s\sqrt{2} \mid s > 0\}$:

(a) Every member is *proportional* <u>*as an ordered pair*</u> to every other member.

(b) The quantity $s\sqrt{2}$ is *proportional* <u>*as a variable quantity*</u> to the quantity s.

Thus statements of the form "*Y* is proportional to *X*" (or "*X* and *Y* are proportional") have two meanings, depending on the context. If *X* and *Y* are ordered pairs, Definition (12a) applies, while if *X* and *Y* are variable quantities, Definition (12b) applies.

Another way of looking both at how (12a) and (12b) are alike and also at how they are different is in terms of the role a *function* of the form $y = kx$ plays in each, stated in terms of the proportionality class $\{(s, s\sqrt{2} \mid s > 0\}$:

[9] These correspond, respectively, to definitions (1) and (2) of the introduction.

(13a) The ordered pairs A and B are *proportional* if A and B are each members of the proportionality class $\{(s, s\sqrt{2} \mid s > 0\}$. In this case a function $B = sA$ maps A to B for some positive real number s. See (12a).

(13b) The variable quantities x and y are *proportional* if the set of ordered pairs (x, y) constitutes the proportionality class $\{(s, s\sqrt{2} \mid s > 0\}$. In this case, the function $y = \frac{a_2}{a_1} x$ maps x to y. See (12b).

In each case, a function of the form $y = kx$ plays a role, but only in (13b) is the parameter k a true constant, or invariant, in the sense that the same k applies for different x and y. The invariant is the constant $\frac{a_2}{a_1}$. In (13a), on the other hand, the parameter s in the function $B = sA$ is not a constant at all, but in general differs for each choice of the two ordered pairs A and B from the proportionality class (13).

3. Proportionality in n-tuples

In this section we extend the characterization of the two kinds of proportionality to the general case of n-tuples. For the most part this is a straightforward generalization of the discussion of ordered pairs in §2.

The main point is that there are two answers to the question of what sorts of mathematical objects X and Y can be in a statement such as "Y is proportional to "X". We can state each of these in terms of the notion of a proportionality class such as

(14) $\{(sa_1, sa_2, ..., sa_n) \mid s > 0\}$.

On the one hand, we have

(15) Definition: Each member of such a proportionality class (14) is *proportional* to every other member.

See Definition (1). Definition (4) is the special case for ordered pairs. Specifically, if $A = (a_1, a_2, ..., a_n)$ is one member of (14), then every other member is *proportional* to A and has the form $(b_1, b_2, ..., b_n) = sA = (sa_1, sa_2, ..., sa_n)$. We can write this relationship succinctly as

(16) $B = sA$ $\qquad\qquad$ $B \propto A$

Here, it is *n–tuples of numbers* that are proportional, where the meaning is that one n–

tuple is a dilated or scaled version of the other n–tuple. In particular, the factor of dilation is a "scale factor". It represents a size relationship between two n–tuples.

On the other hand, consider again the proportionality class (14):

(17) Definition: For each pair i, j in (14), where 1 ≤ i, j ≤ n, we say that the variable quantity sa_i *is proportional to* the variable quantity sa_j.

This states in terms of n-tuples what Definition (8) stated for ordered pairs. The meaning is that ratios $r_{ij} = \frac{sa_j}{sa_i}$ of corresponding terms of different n–tuples in the proportionality class are equal. This can be expressed in the functions

(18) $y = r_{ij}x$

where *x* ranges over the ith terms sa_i in the ordered pairs (sa_i, sa_j) of the proportionality class (14). The ratio is the *constant of proportionality*. It is a "shape factor" representing the relationship between the ith terms and the jth terms of the proportionality class that contains $(a_1, a_2, ..., a_n)$.

In the case of n-tuples, there are n^2 different constants r_{ij} in (18). We note that

(19) $r_{ii} = 1$ for $1 \le i \le n$

(20) $r_{ij} = \frac{1}{r_{ji}}$ for $1 \le i, j \le n$

In summary, in Definition (17), it is sets of variable quantities linked by a function (18) that are proportional, while in Definition (15) it is n-tuples that are proportional.

It is clear that the functions (16) and (18), although they have the same general form $y = kx$, are completely different in what sort of mathematical objects *x* and *y* are. In the function (16), *x* and *y* are n–tuples *A* and *B* of real numbers, whereas in the function (18), *x* and *y* are variable quantities ranging over the positive real numbers.

4. Variability

For any statement of the form "*Y* is proportional to *X*" to make sense, there must be some "variability". This variability shows up in different ways depending on whether it is the sense of *proportional* in Definition (15) or in Definition (17) that is intended. We start with the Definition (15) sense. For example, consider the statement

(21) In similar figures, measures of corresponding parts are proportional.

The "variability" implied in statement (21) is over all positive reals (through the *s* of Definition (15)). But a statement very much like (21) can refer to just *two* similar figures:

(22) In two similar figures, measures of corresponding parts are proportional.

So the variability of *s* over all positive reals has been reduced to two. That is, there are just *two* n-tuples implicit in statement (22) (assuming there are n "parts" in these figures). Since the variability in Definition (15) relies on the notion of "corresponding parts" between two figures, we can't reduce this variability any more. In a single figure, no matter how many "parts" it has, there are no "corresponding" parts, and hence no variability. In an single figure, nothing is proportional to anything else.

Let us turn to the question of variability in the Definition (17) sense of *proportional*. For example, consider the statement

(23) In a set of similar figures, every part is proportional to every other part.

What is the minimum "variability" consistent with the use of the term "proportional"? As above, we must have at least *two* similar figures. Further, we can certainly consider figures with only *two* parts. See example (10) above. This is, in fact, the minimum.[10]

In summary, in this context of similar figures, the minimum of variability for a statement of the form "*Y* is proportional to *X*" is that we need at least *two* figures, each with at least *two* parts. For such a pair, we can make two kinds of statements about proportionality.

(24) In a pair of *two* similar figures, each with *two* parts:

(a) Measures of corresponding parts are proportional.

(b) Each part is proportional to the other.

Since the figures have just *two* parts, we can use the ordered pair definitions of §2. In these statements, (24a) has the Definition (4) sense of *proportional*, while (24b) has the Definition (8) sense of *proportional*.

To be clear about what these statements mean, let us represent the lengths of the two parts in one of the figures by the pair (a_1, a_2). Then the lengths of the corresponding parts of the other figure are given by the pair (ka_1, ka_2) for some $k \geq 0$.

[10] A "figure" might have only one "part". For example, a line segment is a figure. Further, the set of all line segments are similar. However, (23) cannot refer to such a set, since there is no variability within a figure. In terms of (23), there is no "other" part.

By Definition (4), statement (24a) means that the second pair, (ka_1, ka_2), is a multiple of the first pair, (a_1,a_2). This is clearly the case:

(25a) $(ka_1,ka_2)=k(a_1,a_2)$

By Definition (8), statement (24b) means that, looking at both pairs together, the "variable quantity" $[a_1,ka_1]$ consisting of the first terms of these pairs, is a constant multiple of the "variable quantity" $[a_2,ka_2]$ consisting of the second terms of these pairs.[11] So here, the variable quantity sa_1 referred to in Definition (8) for the first term of the pairs has just two elements, $[a_1,ka_1]$. And similarly for the second term. This is the minimum of variability possible.

In summary, by Definition (8), statement (24b) means that $[a_1,ka_1]$ is a "constant multiple" of $[a_2,ka_2]$. And this is true. The "constant" is the ratio $\frac{a_2}{a_1}$:

(25b) $a_2=\frac{a_2}{a_1}\times a_1$ and $ka_2=\frac{a_2}{a_1}\times ka_1$.

It is common to refer to the situation of (24) and (25) by saying that there are four numbers $a_1,a_2,ka_1,$ and ka_2 that are "proportional". The arithmetical relationships that we have stated in (25a) and (25b) are typically stated in terms of a "proportion", meaning an equality between two ratios, often considered as fractions.

The two proportions relevant to this situation are

(26) $\frac{a_1}{ka_1}=\frac{a_2}{ka_2}$

(27) $\frac{a_1}{a_2}=\frac{ka_1}{ka_2}$

Since each equality here is true, these are true proportions. But does (25a) correspond to (26) or (27)? The answer to this is more complicated than one would think. In part it depends on conventions for representing the "ratios" of (26) and (27). If ratios need to be between quantities of the "same kind", then we need to know what this means. It also depends in part on the role of the "constant of proportionality". See (9).

[11] We use square brackets here, […] to group the first terms of a set of ordered pairs together, and similarly for the second terms. We continue to use parentheses (…, …) for the ordered pairs themselves.

We address some of these issues in the next (and final) section.

5. *Interchanging the* **size** *and* **shape** *roles*

Recall statement (5):

(5') In isosceles right triangles, corresponding sides are proportional.

Here the sense of *proportional* is that of definition (4). If we represent two of the sides of isosceles right triangles as the ordered pairs (*leg, hypotenuse*), then every pair in the proportionality class $\{(s,\sqrt{2}s) \mid s>0\}$ is proportional to every other pair. The number *s* here indicates a ***size*** relationship between isosceles right triangles. Every such triangle is a *scaled* or *dilated* version of every other such triangle.

On the other hand, recall statement (10):

(10') In isosceles right triangles, the hypotenuses are proportional to the corresponding legs.

Here the sense of *proportional* is that of Definition (8). The proportional relationship can be expressed in the function $h=\sqrt{2}l$. The quantities *h* and *l* are proportional in the sense that one is a constant multiple of the other. The number $\sqrt{2}$ is the *constant of proportionality*. It indicates a ***shape*** relationship in isosceles right triangles. It has nothing to do with the sort of *scaling* or *dilation* or *size* function of the number *s* discussed under (5) just above.

On the basis of these examples it may be tempting to say:
(a) Any *constant of proportionality* always works like the ***shape*** constant $\sqrt{2}$ in (5) as a case of the Definition (5) sense of *proportional*.
(b) Any ***size*** relationship uses the number *s* in a *scaling* or *dilation* role of the Definition (8) sense of *proportional*.

Still, there is a type of case where these apparently clear and different *size* and *shape* roles seem to change places, and where the constant of proportionality is not a *shape* factor, but rather a *size* factor. This type of case occurs in maps and scale drawings, where what is clearly a *size* factor, the *scale* of the map or drawing, is the constant of proportionality.

To illustrate, consider a case where we have two floor plans of the same room, where the scale of one is, say, 7 times the scale of the other. In this case we certainly say that lengths in the large plan are proportional to corresponding lengths in the smaller plan.

The proportional relationship can be expressed in the function

(28) $y = 7x$

This expresses the fact that lengths in the large plan are 7 times the corresponding lengths in the smaller plan. If there are n different lengths in the plan, then y and x in (28) each can be thought of standing for an n-tuple, and so (28) assumes the character of a scaling function of the general form $y = sx$, where x and y are n-tuples, as in the discussion of Statement (12) above.

However, in the floor plan example, s is definitely **not** a variable that ranges over the positive reals, as is the case with the scaling functions we have discussed so far. Rather, s is a constant 7. It is the constant of proportionality expressing the proportional relationship between corresponding lengths in the two floor plans. Thus proportionality in the case of relationships in maps and scale drawings is rather different from the other cases of proportionality considered earlier.

To make this more clear, consider the simple case of all rectangles where the long side is 7 times the short side. In this similarity class of rectangles, we say that the long sides are proportional to the corresponding short sides, and that (28) expresses this proportional relationship. In (28), x refers to the short side and ranges over the positive reals, while $7x$ refers to the corresponding long side and also ranges over the positive reals.

Formula (28) also describes the fact that in two floor plans of the same room, one 7 times the scale of the other, the lengths in the large plan are proportional to corresponding lengths in the smaller plan. But here, x refers to a *single fixed n-tuple* of lengths. Similarly, y refers to (another) *single fixed n-tuple* of lengths. So x and y are not variables.

Since in this case x and y are not variables, we might wonder where the variability[12] is in the floor plan interpretation of (28). The variability is over the n different lengths represented in the floor plan.

[12] Variability in this sense is discussed in §4.