Proportionality

Definitions

There are several definitions related to proportionality. We will begin with a definition that is typically provided in school textbooks.

Definition 1. A *proportion* is a statement of equality between two ratios. A *ratio* is typically defined in textbooks as a quotient of two real numbers. A more formal statement defines a *proportion* to be two ordered pairs (x_1, y_1) and (x_2, y_2) in $\mathbf{R} \times \mathbf{R}$ with $y_1/x_1 = y_2/x_2$.

Definition 1 involves precisely four quantities; consequently it only captures variation between two possible values of *y* and two possible values of *x*. We are often interested in situations involving more than two values of a variable. The next definition generalizes the notion of proportion to include all pairs of values whose ratios are equal to the same value. An additional artifact of Definition 1 being expressed in terms of a ratio is that the ordered pair $(0,0)$ is excluded from a proportion, thus to generalize to a broader range of variability, it is sometimes beneficial to express the defining equation as a product, $y = kx$, rather than a quotient, $y/x = k$. Making both of these changes thus results in a mathematically different meaning of proportion.

Definition 2. A set of ordered pairs $P = \{(x, y) \in \mathbf{R} \times \mathbf{R}\}$ is a *proportion* if and only if there is a constant $k \in \mathbf{R}$ such that $y = kx$ for all $(x, y) \in P$. Some statements of this definition require *k* be nonzero.

Definition 2 satisfies the conditions of the Bourbaki definition of a function. Most people will be more familiar with an equivalent formulation phrased in terms of the Dirichlet definition of function, emphasizing variation rather than a static set of ordered pairs.

Definition 3. Two varying quantities *y* and *x* are *proportional* if and only if there is a constant $k \in \mathbf{R}$ such that $y = kx$. The constant, *k* is called the *constant of proportionality.* We also say that *y is proportional to x* and write $y \propto x$. Thus if $y \propto x$ with constant of proportionality $k \neq 0$, then $x \propto y$ with constant of proportionality $1/k$. This definition can be given a slightly more formal statement by defining a function *f* to be a *proportional relation* if there is a constant $k \in \mathbf{R}$ such that $f(x) = kx$.

Definition 3 is often preferred when modeling dynamic events since it emphasizes a function relation between the variables *x* and *y*. It may also be presented in the following form stressing the connection to linearity in one dimension.

Definition 4. A function $f : \mathbf{R} \to \mathbf{R}$ is *linear* (*a proportional relation*) if and only if $f(cx) = cf(x)$ for all $c, x \in \mathbf{R}$.

Note 1: The term "linear" is used differently in school mathematics, where any function of the form $f(x) = kx + b$ is considered linear, however such functions may be less ambiguously termed "affine functions." Definition 4 follows the more standard use of the term "linear," and only includes those affine functions with $b = 0$.

Note 2: The general definition of a linear function with arbitrary domain, *V*, includes two parts: i) $f(cx) = cf(x)$ for all $x \in V$ and scalars *c*, and ii) $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all $x_1, x_2 \in V$. Part ii) is unnecessary in the case of a function $f : \mathbf{R} \to \mathbf{R}$ (or on any one-dimensional domain). If $x_1, x_2 \in \mathbf{R}$ and $x_1 \neq 0$ then $x_2 = \frac{x_2}{x_1} x_1$, and a couple applications of Part i) results in Part ii). If $x_1 = 0$, then Part i) implies $f(x_1) = 0$ so again Part ii) holds.

Definition 4 states the property that multiplication by a constant commutes with evaluation of the function. In terms of two proportional quantities *y* and *x*, it says that if *x* is scaled by some constant, then *y* must also be scaled by the same constant.

Definitions 2-4 are mathematically equivalent, but use slightly different language and focus on different characteristics of a proportional relationship. These differences are the basis for different ways of proportional reasoning, the subject of the subsequent entry "Two definitions – Two reasoning patterns."

The quantities in a proportional relationship are often measures of some aspect of a set of things. Thus viewing each quantity as a real-valued function on a set of objects under consideration leads to the following characterization of proportionality.

Definition 5. Let *X* be a set and let $f, g : X \to \mathbf{R}$. Then *f* and *g* are *proportional* if there is a constant $k \neq 0$ such that for all $x \in X$, $g(x) = kf(x)$.

A subsequent essay further develops this final definition, examples, and its relationship to classical and modern notions of proportionality.

Proportional reasoning

Proportional reasoning is a critical area of middle-school mathematics. According to Lesh, Post & Behr (1988), it plays a

> "conceptual watershed" role...at the borderline which separates elementary from more advanced concepts. That is, it is both (1) one of the most elementary higher order understandings and (2) one of the highest level elementary understandings.

It is a capstone of elementary mathematics and a portal to many high-school mathematics topics.

Multiple levels of abstraction in proportional reasoning

Proportional reasoning involves making comparisons between magnitudes which depends on several stages of abstraction. The least abstract concepts are derived through our immediate experience of objects and events and of their directly perceptible properties—e.g., a piece of string and an image of its length, a stone and a feel for its weight, a container holding an amount of water or an event lasting for some duration. Proportional reasoning is several steps removed from these immediate experiences.

First, to discuss ratios derived from objects other than just pure numbers, we need to set out some ideas about measurement. Measurement is the process by which a number is associated with a real or hypothetical magnitude by a process of comparison with a unit. The unit is a magnitude of the same kind as the object measured that acts a standard. For example, when we measure the length of an object in inches, we use a standard inch or several copies of it (e.g., on a ruler) as a unit and count the number of copies and fractional parts of the standard that is required to equal the length of the object. The comparison to the standard unit may be made directly, but often the actual measurement is deduced from information obtained by means other than direct comparison. For example, in surveying, measurements are obtained by using basic facts of geometry. The number that results from the measurement process is called the length, area, volume, weight, duration (elapsed time), etc. according to the kind of measurement taken. This number is meaningful only when the standard unit is identified. This process of quantification produces a number as an object of thought rather than of our immediate experience.

In a second step of abstraction, different magnitudes are compared as ratios, and these comparisons become new objects of thought. A final step involves making comparisons between the comparisons: the relationship between *those two* magnitudes is compared to the relationship between *these two*. Note that this third step of abstraction comparing ratios also requires that both magnitudes be conceived as variable quantities. Likewise, the second step of abstraction comparing magnitudes requires these variables be paired, or to vary in tandem.

This view of proportional reasoning is recognizable in Euclid. (Perhaps it even originates with him.) Euclid defined a ratio is a comparison of magnitudes:

> **Book V, Definition 3.** A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind.

He defined a proportion, in turn as a comparison between two ratios:

> **Book V, Definition 6:** Let magnitudes which have the same ratio be called proportional.

Thus, in the classical definition of proportion we see two clear steps away from the immediate notion of magnitude. The first abstraction is ratio: a comparison between magnitudes objectified. The second is proportion: a comparison between ratios. In fact the concept of proportion captures the entire equivalence class of ratios. It is neither possible to immediately experience all possible pairs of magnitudes nor possible to directly conceptualize all possible comparisons between pairs. Thus conceiving of a proportional relationship necessarily entails abstracting the idea of the equivalence class of pairs with equal magnitude.

Ratios of measured magnitudes

If two magnitudes are measured in the same units, then the *ratio* of one to the other is the measure (number) of the one divided by the measure (number) of the other. For example, if a is the height of Albert in inches and b is the height of Ben in inches, then the ratio of Albert's height to Ben's is a/b . In this case, ratio is understood as a function that is applied to the measure numbers that have been obtained.

Typically, there is a meaningful relationship of some kind between the magnitudes that are compared by forming a ratio. We could calculate the ratio of Albert's height in inches on his 21st birthday to the circumference in inches of the rim of the Liberty Bell, but there would be no obvious meaning in this. On the other hand, comparing Albert's height on his 21st birthday to Ben's on his is clearly part of an extensive and meaningful series of ratios involving the heights of many people. Understanding the context in which ratios are created and the purpose for their use is typically very important in understanding how to use them. There are numerous paradigmatic situations.

Rates

Traditionally, the term "ratio" is reserved for cases where quotients of like measurements are made (as in a comparison of a height in inches to another height in inches). When related magnitudes of different kinds are measured, then the ratio between the resulting numbers may still be meaningful. In this case, the word "rate" is used. For example, the miles travelled on a trip divided by the number of gallons of gasoline used gives the average for the trip of the rate of distance travelled to fuel consumed in miles per gallon. Similarly, the distance travelled on a trip divided by the time of the trip gives the average speed in units of distance per unit time.

When a rate is given, the units of measurement accompany it, since the meaning of the numerical value of the rate is dependent upon the units used. In contrast, the meaning of a ratio between measurements of the same kind and in the same units is independent of the units used. For example, if the ratio of the width to the height of a rectangle is 2 when inches are used as the unit of measure, then the ratio is the same if centimeters are used as the measure. Sometimes, even when two measurements are made using the same units, it is useful to retain a distinction between them for the meaning of the ratio. For example, on a graph we typically think of slope as the ratio of units of vertical change per units of horizontal change. "Canceling" the units to obtain a "dimensionless" quantity for slope makes the meaning less transparent (although the dimensionless aspect does capture the property that the value obtained for slope is invariant under different choices of units).

Conversion factors

If two different units (e.g., inches and centimeters) are available for making the same kind of measurement, then one may be measured with the other. For example, we may measure the length of an inch using a centimeter as a unit. In fact, an inch measures precisely 2.54 centimeters. This number can be used to convert inch measures into centimeter measures; one simply multiplies the inch measure by the number of centimeters in one inch.

When the same feature of an object is measured using different units, the ratio between the two numbers obtained is always the same. For example, when measured in inches a standard piece of letter paper in 8.5 inches wide and 11 inches long. In centimeters, the width is 21.59 and the length is 27.94 centimeters. In both cases, the ratio of the centimeter measure to the inch measure is 21.59/8.5= 2.54 centimeters per inch. This is the number of centimeters in one inch. The ratio of the inch measure to the centimeter measure is 8.5/21.59= 50/127 = .3937…, which is the number of inches in one centimeter.

Two definitions – Two reasoning patterns

There are a variety of proportional reasoning strategies which individuals often use naturally and among which we even subconsciously alternate. Reflecting on these strategies and their relationship to the mathematical structure of proportionality can afford one greater control over these reasoning strategies and allow for easier recognition of others' strategies. First, let's consider two basic reasoning patterns that are closely tied to two different ways of defining a proportional relationship.

Constant Multiple or Ratio. Definition 3 provided above of a proportional relationship states:

> Two varying quantities *x* and *y* are proportional if and only if $y = kx$ for some constant *k* (thus $y/x = k$ whenever $x, y \neq 0$).

Several proportional reasoning strategies involve determining and/or using the value of this constant of proportionality, *k*. Dividing measurements for a volume and corresponding mass of an object composed of a single material to determine its density is an example of finding this constant. Knowing this constant density, one may then determine a mass corresponding to any other volume or determine a volume corresponding to any other mass. Saying that the ratio of the quantities is constant also allows us to express the general relationship without explicit mention of the constant of proportionality as $y_1/x_1 = y_2/x_2$.

Scaling. Proportional relationships may also be defined as linear functions whose domain and range are both real numbers. Definition 4 above characterized proportionality using function notation and the linearity condition in one dimension:

> A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is linear if and only if $f(cx) = cf(x)$ for all $c, x \in \mathbf{R}$.

While this is mathematically equivalent to the constant multiple definition above, it is not conceptually equivalent. Here the focus is on the effect of scaling one variable by some factor, *c*. The linearity condition requires that the other variable must also scale by the same factor. Letting $y = f(x)$, we may also express this by saying if $x \mapsto y$ then $cx \mapsto cy$. In the example of density, knowing one corresponding pair of mass and volume for a substance, we may multiply both quantities by any constant to generate another pair without ever conceiving of the constant of proportionality *k*. Since the scaling between two values of one quantity, say y_1 and y_2, is simply their ratio, saying that two quantities *x* and *y* always scale by the same factor may be expressed by the equality $y_2/y_1 = x_2/x_1$. Notice that this is different from the equality of ratios established from the constant multiple definition above, and reflects a different way of thinking about the relationships between the quantities.

Changes in proportional quantities are proportional.

If two quantities are proportional then their changes are also proportional with the same constant of proportionality. In other words, if there is a constant *k* such that $y = kx$ for all *x* and *y*, then $\Delta y = k\Delta x$ for all corresponding pairs of changes Δx and Δy. This simple fact is the basis of reasoning patterns based on partitioning one quantity and determining the corresponding partition sizes of the other quantity. For example, suppose we determine that a 2.5 cm^3 sample of copper has a mass of 22.4 g but want to mentally compute the mass of a 3.5 cm^3 sample. Reasoning by scaling, we can see that a 0.5 cm^3 sample of copper should have a mass of about 4.5 g (one fifth of the original 2.5 cm^3 is 0.5 cm^3 and one fifth of the original 22.4 g is $20/5 + 2.4/5 \approx 4 + 0.5 = 4.5$ g). So adding 0.5 cm^3 to the volume twice should correspond to adding 4.5 g to the mass twice. Thus a 3.5 cm^3 sample should have a mass of about 31.4 g.

That the changes of proportional quantities are also proportional can sometimes create surprising results. A classic proportional relationship question is to imagine it is possible to wrap a rope snugly around the equator of the earth then ask, "How much would you need to increase the length of the rope in order to lift it one foot from the surface of the earth the entire way around the equator?" Many people who know that the circumference, C, and radius, r, of a circle are related by $C = 2\pi r$ will try to solve a problem like this by looking up the radius of the earth then computing the two appropriate circumferences. A simpler approach is to use the fact that $\Delta C = 2\pi\Delta r$ and note that $\Delta r = 1$ foot, so $\Delta C = 2\pi$ or about 6.3 feet. Since this added length seems insignificant in comparison to the circumference of the earth and since the computation did not take into account the size of the earth, most people find this result difficult to believe.

The converse is not true. In particular, if there is a constant k such that $\Delta y = k\Delta x$ for all pairs of changes Δx and Δy, then $y = kx + b$ for constants k and b. The fact that the changes are proportional is reflected graphically by the fact that the slope k can be computed using any corresponding changes Δx and Δy.

How do we know when two quantities are proportional?

Suppose we record the outdoor air temperature at different times after sunrise. If the temperature 2 hours after sunrise is 70° F, what can we infer about the temperature 4 hours after sunrise? Assuming a proportional relationship results in the nonsensical conclusion that the temperature would be 140° F. Nevertheless, this type of assumption is often made implicitly in situations where the error is less obvious. Ultimately, justification for a proportional relationship requires one to conclude the equivalent of one of the definitions stated above. Examples of reasoning on which such an argument may be based are

i) Scientific theory or a mathematical model: The decay rate of a radioactive material is proportional to the amount, A, i.e., ${}^{dA}\!/_{dt} = kA$, where $k < 0$. One way to reasonably suspect this relationship is to use a scaling strategy to note that if one doubles or triples the amount of radioactive material, you would expect double or triple the number of decays in a given time increment, respectively. Likewise, a simple population model might posit that in any given time frame some fixed (constant) percentage of the population, P, can be expected to be reproducing. This allows us to argue for a constant multiple formulation, that ${}^{dP}\!/_{dt} = nkP$, where k is the fraction of the population reproducing in a unit of time and n is the average number of offspring.

ii) Geometric similarity: Definitions of similarity typically include a statement such as "ratios of lengths of corresponding sides are equal." This implies that the measures of corresponding sides of a collection of similar figures are proportional using the constant ratio definition. Application of theorems that imply similarity may thus also be thought of establishing proportional relationships.

iii) Measurement conversion: Sometimes the relationship between two quantities may be interpreted as using different units to measure the same thing. For example, in a classic Piagetian task, children are asked to guess how high water in a wide cylinder will reach when poured into a second narrower cylinder. Equally spaced marks on each cylinder may be thought of as different units for measuring volume. Thus the conversion factor between units would be constant, giving a constant multiple justification for the proportionality between measurements made in the two cylinders.

iv) Assumption: Often proportional relationships are merely assumed. For example, one may assume that when purchasing some product, the cost and quantity should be proportional. While this assumption is often implicit in school mathematics problems, many real-world pricing schemes allow the consumer to pay less per unit when purchasing larger quantities.

Variation, Correspondence and Equal Partitioning

The constant multiple or ratio and the scaling conditions are the most salient aspects of the definitions of proportionality provided above. There are, however, two other key components: 1) both quantities must be variable and 2) there is a 1-1 correspondence between the variable quantities.

If we are considering constant quantities, say the price that Tom and Cindy paid to add a room to their house and the square footage of that addition, then we could compare the two quantities as a ratio and even conceive of that ratio in meaningful ways, such as the price they paid per square foot. Such a ratio can only become a proportion if we consider these quantities in a way that they are allowed to vary. In this remodeling example, perhaps we could consider the cost and square footage of all home additions in some region during a given year. Due to variations in pricing, these quantities are unlikely to be proportionally related, unless perhaps we consider the price of additions built by a single contractor who charges a fixed rate per square foot.

A second important aspect of the definition of proportionality is a 1-1 correspondence between the two variable quantities. In the case of the contractor who charges a fixed rate per square foot, each price has a unique cost associated with it, and reciprocally, each cost has a unique corresponding price. In the more general context considering all additions in a region, we would not expect such a relationship. This 1-1 correspondence is implicit in the way we write and say things about proportional relationships. For example, if we write $y = kx$, we are conceiving of *y* and *x* as paired variables and it is the specific value of *y* which corresponds to *x* that is equal to kx. If we write $y_1/x_1 = y_2/x_2$, the use of subscripts make this correspondence slightly more explicit by denoting various values differently (as opposed to the blanket use *y* and *x* for all values). Similarly, writing something like $\Delta y/\Delta x = k$ means that we are considering the change in *x* that corresponds to that particular change in y. In other words, if $y = kx$ generally, then we may call out specific paired values $y_1 = kx_1$ and $y_2 = kx_2$ whose differences are $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$.

Finally, we often want to consider various ways of partitioning quantities by equal increments. One important partitioning is by portions that are each the size of the unit of measurement. Each unit represents equal quantities, and the number of partitions is the magnitude of the measured quantity. When considering two related quantities, we often partition one into a certain number of units then partition the other into the same number of parts – possibly not units. In this way each unit-sized portion of the first quantity corresponds exactly to a particular fixed size interval of the second quantity. For example the number of days in one orbit of the earth around the sun is about 365.26 days in which it travels about 9.4×10^8 km. If we imagine this time duration partitioned into one-day time intervals, then partition the distance into 365.26 equally-sized lengths, each one will be 2.6×10^6 km. Thus, the average speed of the earth around the sun is about 2.6×10^6 km/day. Although the earth does not travel at a constant speed, the variation is far smaller than the level of accuracy captured by the two-digit quantities used here. Regardless, we see that this view of equal and corresponding partitioning is important for conceptualizing the meaning of average rate of change.