Ratio and Proportion in Euclid

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August 10, 2008

1. Introduction

The topic of ratio and proportion has been part of school mathematics since modern schooling began. It includes the advanced arithmetic and rudimentary algebra used in dealing with quantities x and y that are related by an equation of the form $y = kx$ and it has numerous applications in science and commerce.

Much of what is taught in school comes from two sources. The arithmetical root is the ancient procedure known as the "Rule of Three", used for solving problems such as finding the price of one quantity of some commodity if the price of another quantity of the same commodity is known. The geometric root is Book V of Euclid's *Elements*, where Euclid develops the concepts of ratio and proportion for (unmeasured) geometric magnitudes. The words "ratio" and "proportion" are in fact derived from the Latin words that Cicero suggested as renderings for Euclid's *logos* and *analogia*.[1]

Book V contains 18 definitions and 25 propositions on the theory of ratio and proportion. Heath, in his commentary, writes:

> The anonymous author of a scholium to Book V, who is perhaps Proclus, tells us that 'some say' this Book, containing the general theory of proportion which is equally applicable to geometry, arithmetic, music, and all mathematical science, 'is the discovery of Eudoxus, the teacher of Plato.'

What is this theory, and how does it relate to the modern conceptual system of ratio and proportion? The chief concerns of Euclid's theory, as we shall see, are very different from those that arise when dealing with measured quantities. Euclid's theory, in fact, includes (implicitly) an analysis of the measurement process itself, and in this it goes far deeper than anything we find in typical schoolbook expositions of ratio and proportion. In the present note, we will present an analysis of Book 5 that clarifies these matters.[2]

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[1] See Alexander John Ellis. (1874). "Euclid's conception of ratio and proportion," in *Algebra Identified with Geometry*. London: C. F. Hodgson & Sons.

[2] Other useful commentaries on Euclid but to address issues that are associated with teaching include Ellis, *op. cit.* and Augustus De Morgan. (1836). *The Connection of Number and Magnitude: An Attempt to Explain the Fifth Book of Euclid*. London: Taylor and Walton. Reprinted by Kessinger Publishing, 2004.

2. Contrasts between Ancient and Modern Mathematics

The mathematics of ancient Greece differs from modern mathematics in profound ways. Number systems were more rudimentary, and numbers and numerical measurements played a very limited role in Euclidean geometry.

Aristotle, in his work *Categories* (written about 50 years before Euclid), classified quantity as "either discrete or continuous." The primary example of a discrete quantity was number, and number meant a collection of units. For the Greeks, the numbers were the positive integers—nothing more. They had no "real number system" and no "number line". Among the continuous quantities, one found the objects of geometry: segments, planar regions, and other things that Euclid later referred to as "magnitudes". They were not numbers, and had no numbers attached. Ratio, as treated in Book V, is a relationship between magnitudes. As such it is an abstraction outside the realm of number.

A passage written by Newton in the 1660s shows that by then, the Greek conception of number had been turned on its head:

> *By* Number *we understand,* not so much a Multitude of Unities, as *the abstracted Ratio of any Quantity, to another Quantity of the same Kind, which we take for Unity.* And this *is threefold; integer, fracted, and surd: An* Integer, *is what is measured by Unity; a* Fraction*, that which a submultiple Part of unity measures; and a* Surd, *to which Unity is incommensurable.*[3]

The new and broader conception of number predated Newton, of course, but this is not the place to explore its history. It suffices to note how fundamentally different our conception of number is from the ancient one. For us, the counting numbers are merely the most rudimentary level of a hierarchy of ever more general systems, culminating with the real numbers (or beyond). Our richer number system provides us with the machinery for treating ratio and proportion in a much different manner from the ancient Greeks.

A second major difference between ancient and modern mathematics concerns the immediacy of the objects, as we see in the following passage from David Fowler's reconstruction of the ancient Greek mathematical mentality:

> Greek mathematicians seemed to confront directly the objects with which they were concerned: their geometry dealt with the features of geometrical thought experiments in which figures were drawn and manipulated, and their *arithmetike* concerned itself ultimately with the evident properties of numbered collections of objects. Unlike the mathematics of today, there was no elaborate conceptual machinery, other than natural language, interposed between the mathematician and his problem. Today we tend to turn our geometry into arithmetic, and our arithmetic into algebra so that, for example, while Elements I.47: "In right-angled triangles the square on the side subtending the right angle

[3] *Universal Arithmetic* (translated from the Latin by Ralphson), London 1769, page 2. The italicization is from the original.

> is equal to the squares on the sides containing the right angle" means literally to Euclid, that the square can be cut in two and manipulated into other squares…, the result is now usually interpreted as: "$p^2+q^2 = r^2$," where we now must explain just what the 'p's, 'q's and 'r's are and how they can be multiplied and added. To us, the literal squares have been replaced by some abstraction from an arithmetical analogy.[4]

The point that Fowler is making is not about a difference in depth or abstraction, but rather about how pervasively and automatically we moderns translate geometry into numerical and algebraic language and deal with geometric facts and ideas in these terms. We take for granted a much more immediate connection between numbers and things—a broad range of things—than the ancients ever conceived. For us, virtually everything has a numerical measure attached: distance, mass, time, price, academic performance. Quantifiction is a mark of the mentality our time.[5]

We think of a ratio as a number obtained from other numbers by division. A proportion, for us, is a statement of equality between two "ratio-numbers". When we write a proportion such as

$$\frac{a}{b} = \frac{c}{d},$$

the letters refer to numbers, the horizontal bars are operations on numbers and the expressions on either side of the equals sign are numbers, or at least become numbers when the numerical values of the letters are fixed.

For the Greeks, this was not the case. When Euclid states that the ratio of A to B is the same as the ratio of C to D, the letters A, B, C and D do not refer to numbers at all, but to segments or polygonal regions or some such magnitudes. The ratio itself, according to Definition V.3, is just "a sort of relation in respect of size" between magnitudes. Like the definition of "point", this tells us little; the real meaning is found in the use of the term. It is in the rules for use that we find the amazing conceptual depth of the theory.

The definition that determines how ratios are used is V.5. This tells us how to decide if two ratios are the same. The key idea is this. If we wish to compare two magnitudes, the first thing about them that we observe is their relative size. They may be the same size, or one may be smaller than the other. If one is smaller, we acquire more information by finding out how many copies of the smaller we can fit inside the larger. We can get even more information if we look at various multiples of the larger, and for each multiple determine how many copies of the smaller fit inside. So, a ratio is implicitly a comparison of all the potential multiples of one magnitude to all the potential multiples of the other. (Two magnitudes are incommensurable exactly when no multiple of one is ever exactly equal to any multiple of the other.) To compare two ratios, $A{:}B$ and $C{:}D$, then, we ought to be prepared to compare the array of all multiples of the first pair with the array of all

4 David Fowler. (1999). *The Mathematics Of Plato's Academy: A New Reconstruction. Second Edition.* Oxford: Clarendon Press. Page 20.

5 Theodore M. Porter. (1995). *Trust in Numbers: The Pursuit of Objectivity in Science and Public Life.* Princeton: Princeton University Press.

multiples of the second. Suppose that we find that for each pair of positive integers (m, n), mA exceeds nB exactly when mC exceeds nD. This, according to Euclid's definition, is when we say the ratios are the same.

3. What properties do Euclidean magnitudes posess?

As we have said, a ratio is a relationship between magnitudes. To understand Euclid's theory fully, then, we need to know what magnitudes are. Segments, polygonal regions in the plane, volumes and angles were clearly included, but beyond these kinds, we do not know what other things Euclid might have viewed as magnitudes. Fortunately, it is not hard to infer from Book V the properties that a class of things must have and what the operations we must be able to perform on them in order to be able to apply the theory. There are clear indications in his writings of the following:

A) Magnitudes are of several different kinds, *e.g.*, segments, polygonal regions, volumes, angles—possibly weights and durations.

B) Given two magnitudes of the same kind, exactly one of the following is true: a) they are the same with respect to size (we say "equivalent"), b) the first exceeds the second or c) the second exceeds the first. (We call this the law of *trichotomy* for magnitudes.)

C) Magnitudes of the same kind may be added to one another—or a given magnitude may be added to itself one or more times—to yield a new magnitude of the same kind that is larger than any summand. No matter how the addition is performed, the outcome has the same size. *Furthermore*, given two magnitudes of the same kind but of different size, a part of the larger equivalent to the smaller may be removed, and no matter how this removal is done, the remainders are equivalent.

D) The relationships of equivalence and of excess are compatible with addition and subtraction in the sense that if a equivalent magnitudes are added to (or taken from) each of two others, the resulting magnitudes will be in the same relation as the originals.[6]

For Euclid, addition or subtraction of magnitudes was a concrete process. In the case of segments, addition and subtraction are described in Book I, Propositions 2 and 3. The addition of polygonal regions occurs in Book I beginning in the proof of Proposition 35[7] and continues through the the proof of the Pythagorean Theorem. As a matter of fact, Euclid's proof of the Pythagorean Theorem is itself an explicit procedure for slicing up two square regions and rearranging the parts to make a third square region which is their sum. In general, there may be numerous ways to add two magnitudes. For example, when two polygonal regions are added, they may be cut into pieces and reassembled in many different ways. Euclid took for granted that when addition of the same magnitudes is performed in two different ways, the results will always be equivalent, even if the relations between the assembled parts in each result are different.

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[6] That A)—D) characterize the magnitudes as conceived by Euclid is confirmed by I. Grattan-Guinness. (1996). Numbers, Magnitudes, Ratios, and Proportions in Euclids Elements: How Did He Handle Them? *Historia Matematica* 23, 355-375.

[7] "Parallelograms which are on the same base and in the same parallels equal one another."

Henceforth, we will use capital letters A, B, *etc.* to stand for magnitudes. We write $A \simeq B$ to mean that A and B are of the same kind and are equivalent. $A \prec B$ means that A and B are of the same kind and B is larger, and $A \preceq B$ means that either $A \prec B$ or $A \simeq B$. $A + B$ denotes a (not "the") result of adding A and B in some way. mA denotes a sum of m copies of A, assembled in some way, and if $B \prec A$, then $A - B$ denotes a difference. Item D) says that if $A \preceq B$ and $C \simeq C'$, then $A + C \preceq B + C'$.

For the next section, it will be convenient to have a concept at our disposal that Euclid never named.

Definition. We shall say that two magnitudes are *in the same archimedean class* if either they are equivalent or else there is a multiple of the smaller that exceeds the larger. We call a kind of magnitudes archimedean if all the magnitudes in it are in the same archimedean class, or equivalently, there are no infinitesimals. (Note that all magnitudes are positive; there is no "zero" magnitude.)

4. Ratio

Book V introduces ratio with the following definition:

> **Definition 3.** A ratio is a sort of relation in respect of size between two magnitudes of the same kind.

As we have mentioned, this definition tells us little. We need to wait to see the meaning. Nonetheless, it does tell us that there is a *something* that we may associate with some pairs of magnitudes. In these notes, we'll use the symbol $(A\!:\!B)$ to stand for the ratio of A to B, if they have a ratio.

> **Definition 4.** Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.

Definition 4 has been much discussed by commentators. For Heath, it raises the question of whether Euclid believed that all the magnitudes of a given kind were in the same archimedean class, or whether he allowed that some kinds might contain magnitudes that were infinitesimal with respect to others in the same kind. As pointed out by David Joyce,[8] horn angles—*i.e.*, the angles formed at the point of tangency of two circles or of a circle and a line—provide an interesting case. Euclid might have viewed these as angles that are infinitely small with respect to any angle between straight lines. However, in the proof of Proposition 8 of Book V, Euclid takes the difference between two arbitrary magnitudes comparable to a third, and asserts that the difference is also comparable with the third. Here, he is implicitly using the assumption that there are no infinitely small magnitudes. Also, as we shall show later, Proposition 9 is false if infinitely small magnitudes are admitted. De Morgan suggested that Definition 4 was intended as a way to test whether two magnitudes were of the same kind. It would exclude a ratio of a segment to an angle. It would also exclude a ratio of a segment to a region, since no number of segments can cover a region. A third possibility is that Definition 4 says just what it says. Euclid might have perceived the possibility of infinitely small magnitudes, or of magnitudes like horn angles

[8] See: `http://aleph0.clarku.edu/~djoyce/java/elements/bookV/defV4.html`

for which there is no obvious addition, and felt the need to specify explicitly what pairs of magnitudes the theory applied to.

The critical definition in the Euclidean theory of proportion is:

> **Definition 5.** Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

In Definition 5, the magnitudes in the first ratio must be of the same kind, as must the magnitudes in the second, but the two magnitudes in the first ratio may be of a different kind from the two magnitudes in the scond. Indeed, in the first application of the theory in Book VI, ratios of segments are compared to ratios of regions. The possibility of relating ratios of one kind of magnitude to ratios of another is one of the truly deep aspects of the theory.

Definition 5 can be translated into more modern language as follows.

Definition 5′. We say that $(A{:}B)$ *is equivalent to* $(C{:}D)$, and write $(A{:}B) \simeq_r (C{:}D)$, if for all positive integers n, m:

$$m\,B \prec n\,A \Leftrightarrow m\,D \prec n\,C \quad \& \quad m\,B \simeq n\,A \Leftrightarrow m\,D \simeq n\,C\ .$$

Note that the "alike fall short of" case is taken care of by trichotomy for magnitudes. It is clear that $\simeq_r$ is an equivalence relation.

> **Definition 6.** Let magnitudes which have the same ratio be called proportional.

Definition 6 supplies a name for the sameness defined in Definition 5 and extends the scope, so we can speak of numerous ratios all at once. If the ratios $(B_i{:}A_i)$ are equivalent, then we say that the B_i are proportional to the A_i. This language is used in Book V, Proposition 12.

> **Definition 7.** When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a greater ratio to the second than the third has to the fourth.

Definition 7 can be translated into our language as follows:

Definition 7′. We write $(C{:}D) \prec_r (A{:}B)$ if for some positive integers m, n:

$$m\,B \prec n\,A\ \&\ m\,D \succeq n\,C\ .$$

5. The Content of Book V

In this section, we translate the propositions in Book V into the language we have introduced. When a statement and its converse are in two different propositions, we have combined them. In several cases, we have eliminated redundant variables (as in the illustration of Proposition 1, above) or used other features afforded by modern notation to streamline the statements.[9]

Propositions 1, 2, 3, 5 and 6 concern multitudes of magnitudes, but not ratios. Each of the following should be read as an assertion about arbitrary positive integers m and n and arbitrary magnitudes A_i, B_i of a single kind.

1. $m\,(A_1 + A_2 + \ldots + x_n) \simeq m\,A_1 + m\,A_2 + \ldots + m\,A_n$.
2. $(m+n)A \simeq m\,x + n\,A$
3. $m\,(n\,A) \simeq (m\,n)\,A$
5. If $B \prec A$, $m\,(A - B) \simeq m\,A - m\,B$
6. If $n < m$, $(m - n)\,A \simeq m\,A - n\,A$

Propositions 7, 8, 9, 10, 11, 13, 14, 16, 20, 21, 22,and 23 concern equivalence and order of ratios, but not addition or subtraction of magnitudes. Propositions 20 and 21 are preliminary to 22 and 23, so we omit them.

11. Transitivity. If $(A_1 : A_2) \simeq_r (B_1 : B_2)$ and $(B_1 : B_2) \simeq_r (C_1 : C_2)$, then $(A_1 : A_2) \simeq_r (C_1 : C_2)$.
13. Substitution. If $(A_1 : A_2) \simeq_r (B_1 : B_2)$ and $(B_1 : B_2) \succ_r (C_1 : C_2)$, then $(A_1 : A_2) \succ_r (C_1 : C_2)$.

7 & 9. Equivalent magnitudes ⇔ equivalent ratios. $A \simeq A' \;\Leftrightarrow\; (A : B) \simeq_r (A' : B) \;\Leftrightarrow\; (B : A) \simeq_r (B : A')$.

8 & 10. Larger magnitudes ⇔ larger/smaller ratios. $A \prec A' \;\Leftrightarrow\; (A : B) \prec_r (A' : B) \;\Leftrightarrow\; (B : A) \succ_r (B : A')$.

7.Cor. Inverse ratios. $(A : B) \simeq_r (C : D) \;\Leftrightarrow\; (B : A) \simeq_r (D : C)$.

14. If $(A : B) \simeq_r (A' : B')$ and $A' \succ A$, then $B' \succ B$.
16. Alternate proportions. If $(A : B) \simeq_r (C : zD)$, then $(A : C) \simeq_r (B : D)$.
22. Ratios ex aequali. If $(A_1 : A_2) \simeq_r (B_1 : B_2)$, $(A_2 : A_3) \simeq_r (B_2 : B_3)$, …, and $(A_{n-1} : A_n) \simeq_r (B_{n-1} : B_n)$, then $(A_1 : A_n) \simeq_r (B_1 : B_n)$.
23. Perturbed ratios ex aequali. If $(A : B) \simeq_r (B' : C')$ and $(B : C) \simeq_r (A' : B')$, then $(A : C) \simeq_r (A' : C')$.

Comment. All of these statements have clear analogies with properties of real numbers that follow easily from properties of multiplication and division. Consider 23, for example. The hypothesis corresponds to saying $\frac{a}{b} = \frac{b'}{c'}$ and $\frac{b}{c} = \frac{a'}{b'}$. From this, we get $\frac{a}{b}\frac{b}{c} = \frac{b'}{c'}\frac{a'}{b'}$, and from this, we get $\frac{a}{c} = \frac{a'}{c'}$, corresponding to the conclusion.

The remaining propositions concern addition.

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[9] I have been aided by the transcriptions made by David Joyce:

`http://aleph0.clarku.edu/~djoyce/java/elements/elements.html`

4 & 15. $(A{:}B) \simeq_r (m\,A{:}m\,B)$.

12. If $(A_1{:}B_1) \simeq_r (A_2{:}B_2) \simeq_r \cdots \simeq_r (A_n{:}B_n)$, then $(A_1{:}B_1) \simeq_r \big((A_1 + A_2 + \cdots + A_n){:}(B_1 + B_2 + \cdots + B_n)\big)$.

17 & 18. $\big((A + B){:}B\big) \simeq_r \big((C + D){:}D\big) \;\Leftrightarrow\; (A{:}B) \simeq_r (C{:}D)$.

19. $\big((D + A){:}(B + C)\big) \simeq_r (D{:}B) \;\Leftrightarrow\; \big((D + A){:}(B + C)\big) \simeq_r (A{:}C)$.

24. If $(A{:}C) \simeq_r (A'{:}C')$ and $(B{:}C) \simeq_r (B'{:}C')$, then $\big((A{+}B){:}C\big) \simeq_r \big((A'{+}B'){:}C'\big)$.

25. If $(D{:}A) \simeq_r (B{:}C)$ and D is the greatest of the four magnitudes while C is the least, then $D + C \succ A + B$.

Like the statements in the second group, all these statements have obvious analogies in the addition of fractional expressions involving real numbers. Of these statements, the most interesting are 24 and 25. 24 is a step toward defining addition for ratios.

6. Understanding Euclidean ratio with modern tools

In this section, we explore the implications of Euclid's assumptions using modern notation. Obviously, this cannot represent a literal line of reasoning that the Greeks might have followed. However, this can reveal structural properties of the mathematical universe that Euclid worked in, and reveal features of it that he might somehow have sensed. The situation is like that with non-Euclidean geometry. It is unlikely that Euclid knew the independence of the Parallel Postulate. But until *we* were aware of this fact, mathematicians could reasonably speculate that Euclid might have lacked the ingenuity to find a subtle proof. The discovery of non-Euclidean geometries gives us insight into Euclid's work, suggesting that the choice of axioms was based on a deeper mathematical sensitivity than had previously been suspected. Note that we do not attempt to identify magnitudes or ratios with any modern mathematical structures, and when we prove facts about magnitudes and ratios, we use only the assumptions we have stated and the kinds of arguments that Euclid might have presented, up to the point where we make a link to a modern idea.

With each ratio $(A{:}B)$, we can associate two sets of positive rational numbers:

$$L(A{:}B) := \Big\{ \frac{m}{n} \mid m\,B \prec n\,A \Big\}. \tag{1}$$

$$E(A{:}B) := \Big\{ \frac{m}{n} \mid m\,B \simeq n\,A \Big\}. \tag{2}$$

The ancients would never have thought of a set like $L(A{:}B)$. First, there is fact that they did not have rational numbers. This might not have been insurmountable—consider the pairs (m, n) such that $m\,B \prec n\,A$. A greater problem might have been their readiness to allow the existence of an infinite collection. In writing about infinity in the *Physics*, Aristotle explicitly denies the existence of an infinitely extended line, and goes on more generally to say that infinity is never actualized, but is just the potential for unlimited growth, addition or other process. He does not say anything explicit about sets (infinite or otherwise), but it seems unlikely that he would have found the idea of gathering an infinite number of number pairs into one completed collection any less objectionable than an infinite line. As we have said, however, we are not attempting to reconstruct ancient thought, but rather to look at the *objects* of ancient thought with the instruments of modern mathematics.

Definitions V.5 and V.7 can be expressed as follows:

Fact 1.
i) $(B_1 : A_1) \simeq_r (B_2 : A_2) \quad \Leftrightarrow \quad L(B_1 : A_1) = L(B_2 : A_2)$ & $E(B_1 : A_1) = E(B_2 : A_2)$.
ii) $(B_1 : A_1) \prec_r (B_2 : A_2) \quad \Leftrightarrow \quad L(B_1 : A_1) \subset L(B_2 : A_2)$ & $L(B_1 : A_1) \neq L(B_2 : A_2)$.

We can derive a number of additional facts:

Fact 2. *$E(A : B)$ never contains more than one element. It may be empty.*

Proof. Suppose $m\,B \simeq n\,A$ and $m'\,B \simeq n'\,A$. Then $mn'\,B \simeq nn'\,A$ and $nm'\,B \simeq nn'\,A$. Thus, $mn'\,B \simeq nm'\,B$. If $mn' \neq nm'$, then one is smaller. Say, $mn' < nm'$. Removing $mn'\,B$ from the quantities on both sides of $mn'\,B \simeq nm'\,B$ leaves nothing on the left, but a positive multiple of B on the right. This is absurd, so $mn' = nm'$, and thus $m/n = m'/n'$. To see that $E(A : B)$ may be empty, let B be the side of a square and A its diagonal./////

Fact 3. *If $m'/n' < m/n \in L(A : B)$ then $m'/n' \in L(A : B)$.*

Proof. Suppose m'/n' and m/n as hypothesized. Then $m'n < mn'$. Since $m\,B \prec n\,A$, $mn'\,B \prec nn'\,A$, so $m'n\,B \prec nn'\,A$, and so $L(A : B)$ contains $m'n/(nn') = m'/n'$. /////

Fact 4. *Let A and B be magnitudes that have a ratio. Suppose that for all m and n, if $m\,B \prec n\,A$ then $n\,A - m\,B$ and B have a ratio.*[10] *Then $L(A : B)$ has no largest element.*

Proof. Suppose $m/n \in L(A : B)$. Then $m\,B \prec n\,A$, so by assumption (and Definition 4), there is k such that $B \prec k\,(n\,A - m\,B)$. Then, $(km + 1)\,B \prec kn\,A$, and accordingly, $(km + 1)/(kn) \in L(A : B)$. /////

From these facts, we can draw a number of interesting conclusions about Eulcid's theory. Since $L(A : B)$ is an interval with left endpoint 0, it is clearly the case that between two ratios, *at most* one of the relations $\prec_r$, $\sim_r$ and $\succ_r$ may hold. This verifies that certain problems noted by Heath in his commentary on Proposition 10, and related problems noted by DeMorgan, are indeed merely omitted arguments. It is also clear that each of the relations between ratios $\simeq_r$, $\prec_r$ and $\preceq_r$ is transitive. This is good news; the theory is tighter than Euclid's exposition.

If any kind of magnitude contains infinitesimals, there are problems over and above those we have already noted in the proof of Proposition 8:

- Definitions 5 and 7 behave oddly. For example, suppose that C is infinitely smaller than A. Then $L(A + A + C : A + C) = L(A + A : A) = [0, 2)$ but $E(A + A + C : A + C) = \emptyset \neq \{2\} = E(A + A : A)$. Thus, none of the relations $\prec_r$, $\simeq_r$ or $\succ_r$ holds between these sets. Trichotomy fails for ratios. One way to repair this problem would be to weaken Definition 5 by omitting the requirement involving $E(B : A)$. With this

[10] Note that this is similar to the unstated assumption used in the proof of Proposition 8, which we commented on above.

altered definition, the equivalence classes of ratios would behave as the set of intervals $[0, a)$ or $[0, a]$ in $\mathbb{Q}_{\geq 0}$, ordered by inclusion. Yet another option would be to define two ratios to be equivalent if the associated intervals have the same least upper bound.

- Proposition 9 asserts that two magnitudes with the same ratio to a third are equivalent. This is false if there are infinitely small magnitudes, regardless of any of the version of Definition 5 we select from among those we've considered. For suppose that C is an infinitely smaller than A. Then $(A + C + C : A) \sim_r (A + C : A)$ by any version of Definition 5, but $A + C + C$ is not equivalent to $A + C$. Thus, infinitesimals are incompatible with Proposition 9.

Recall that we say that a kind of magnitudes is *archimedean* or *has no infinitesimals* if there is only one archimedean class in the kind. If the Euclidean theory is restricted to archimedean kinds, then by Fact 4, $L(B\!:\!A)$ never has a largest element, and $E(B\!:\!A)$ is nonempty if and only if $\rho(B\!:\!A) \in \mathbb{Q}$. In this case, we have an order-preserving embedding of the set of all Euclidean ratios in the set of positive reals. In particular, a) trichotomy for ratios is true, b) to any fixed magnitude, different magnitudes have different ratios (so no problem with Proposition 9 arises) and c) the problem in the proof of Proposition 8 that we alluded to in section 4 in the discussion of Definition 4 is resolved. With the exclusion of infinitesimals, the import of Propositions 7 through 10 is: for any magnitude C, $A \mapsto (A\!:\!C)$ is a $\preceq$-preserving map from the set of magnitudes of the same kind as C to the ratios, and the fibers of this map are equivalence-classes of magnitudes.

These considerations lend weight to the hypothesis that Euclid's theory of ratio and proportion was conceived as theory of archimedean kinds. Euclid might have wanted to leave open the potential of application to comparability classes within non-archimedean kinds, provided ratios were formed within a single archimedean class. He would have had to change some features of the theory to do this, but it would not have been an unreasonable hope. The modern theory of ordered groups shows that such a theory could be developed meaningfully.[11]

7. Relations of Euclid's theory to algebra

Throughout this section, $\mathcal{A}$ will be an archimedean kind of magnitudes and A and U will be magnitudes in $\mathcal{A}$. As we have remarked, $\mathcal{A}$ has an addition and an order that satisfy several axioms. How closely does $\mathcal{A}$ resemble our system of real numbers? Hölder showed that the size-equivalence classes of $\mathcal{A}$ can be embedded in the real numbers in a way that preserves addition and order.[12] In fact, the map that accomplishes this uses the construction of the "lower cut," $L(A\!:\!U)$ that we introduced in the previous section.

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[11] See the discussion of Hahn's representation theory in any textbook on ordered groups.

[12] In an influential paper published in 1901, Otto Hölder set out an abstract mathematical theory of measurement: O. Hölder, *Die Axiome der Quantität und die Lehre vom Mass*, Berichte über die Verhandlungen der Königlichen Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematische-Physische Classe, vol. 53 (1901), pages 1–64. In the process, he made many deep observations about Euclid's theory of ratio and proportion and its relation to the theory he was developing. Hölder's axioms for "measurable magnitudes" included many of the same assumptions we have stated concerning Euclidean magnitudes.

As preparation for proving Hölder's result, we give a lemma. We have shown previously that $L(A{:}U)$ has no largest element. The following lemma strengthens this.

Lemma. *If $q \in L(A{:}U)$ and d is a positive integer, then there are positive integers m and n such that $q = m/n$ and $(m+d)/n \in L(A{:}U)$.*

Proof. There are integers μ and ν such that $q = \mu/\nu$ and $\mu U \prec \nu A$. For some integer k, $dU \prec k(\nu A - \mu U)$. Thus, if we set $m = k\mu$ and $n = k\nu$, we have $q = m/n$ and $(m+d)U \prec nA$. /////

The proof of Hölder's theorem resembles the proof of the continuity of addition, but we present it in a language that, aside from mentioning rational numbers, uses only language that could have appeared in Book V. When we say that a set A of rationals is *cofinal* in another set B of rationals, we mean that every element of B is equalled or exceeded by an element of A.

Theorem. *Suppose $U, A_1, A_2 \in \mathcal{A}$. Then*

$$L(A_1 + A_2{:}U) = L(A_1{:}U) + L(A_2{:}U).$$

Thus, $A \mapsto \sup L(A, U)$ is a $+$-preserving map from $\mathcal{A}$ to the positive real numbers.

Proof. Let $A := A_1 + A_2$, $L := L(A:U)$, $L_1 := L(A_1:U)$ and $L_2 := L(A_2:U)$. By Facts 3 and 4, it suffices to show that L and $L_1 + L_2$ are cofinal in one another. Claim 1: L is cofinal in $L_1 + L_2$. Suppose $m_i/n_i \in L_i$, $i = 1, 2$. Without loss of generality, $n_1 = n_2 = n$. Then $m_1\,U \prec n\,A_1$ and $m_2\,U \prec n\,A_2$, so $(m_1 + m_2)\,U \prec n\,(A_1 + A_2)$, and so $(m_1 + m_2)/n \in L$. This proves Claim 1. Claim 2: $L_1 + L_2$ is cofinal in L. Suppose $q \in L$. By the lemma, pick positive integers m and n such $q = m/n$ and

$$(m+2)\,U \prec n\,A. \tag{1}$$

Pick m_i, $i = 1, 2$, such that

$$m_i\,U \prec n\,A_i \preceq (m_i + 1)\,U. \tag{2}$$

Adding the two equations for $i = 1, 2$, we get:

$$n\,A \preceq (m_1 + m_2 + 2)\,U. \tag{3}$$

From (1) and (3), $(m+2)\,U \prec (m_1 + m_2 + 2)\,U$, so $m < m_1 + m_2$. By (2), $m_i/n \in L_i$. Thus:

$$\frac{m}{n} < \frac{m_1 + m_2}{n} \in L_1 + L_2.$$

One of the main results of Hölder's work is the theorem that every archimedean totally-ordered group is a subgroup of the additive reals. This is one of the pillars supporting the modern theory of ordered groups. A modern statement and proof of Hölder's Theorem may be found on page 48 of: A. Bigard, K. Keimel and S. Wolfenstein. (1977). *Groupes et Anneaux Réticlés*, Springer Lecture Notes in Mathematics, vol. 608.

This proves Claim 2. /////

Some Euclidean magnitudes can be multiplied, in a sense, but multiplication changes the kind. The product of two segments, for example, is a rectangle. This form of multiplication plays an important role in Book II, at the heart of the "geometric algebra" presented in that book. When multiplication of magnitudes is possible, multiplication of ratios can be defined. A simple form of this idea is the statement that if A, B and C are segments and AC and BC are the rectangles with bases A and B and common height C, then $(A{:}B) \simeq_r (AC{:}BC)$. The following theorem extends this thought. The proof that I present is based on the standard δ-ϵ proof of the continuity of multiplication: given numbers a_1 and a_2 and $\epsilon > 0$, find δ_1 and δ_2 such that $\{\, x_1\, x_2 \mid \delta_i > |a_i - x_i|, i = 1, 2 \,\} \subseteq (a_1\, a_2 - \epsilon, a_1\, a_2 + \epsilon)$. Note that δ_1 and δ_2 are dependent not only on ϵ, but also on a_1 and a_2, so the proof has a layer of complexity that was unnecessary in the proof of Hölder's theorem, above. Additional complexity comes from the fact that the proof is phrased in terms of integer multiples of magnitudes. Perhaps the most significant application of this theorem is the "change of unit" theorem, which we state as a corollary.

Theorem. *Suppose $A_1, A_2, U_1, U_2 \in \mathcal{A}$. Then*

$$L(A_1{:}U_1) \cdot L(A_2{:}U_2) = L(A_1A_2{:}U_1U_2).$$

Proof. We show first that $L(A_1A_2{:}U_1U_2)$ is cofinal in $L(A_1{:}U_1) \cdot L(A_2{:}U_2)$. In fact, we show that $L(A_1{:}U_1) \cdot L(A_2{:}U_2) \subseteq L(A_1A_2{:}U_1U_2)$. Suppose $(m_i/n_i) \in L(A_i{:}U_i)$, $i = 1, 2$. Then $m_iU_i \prec n_iA_i$, $i = 1, 2$. Thus $m_1m_2U_1U_2 \prec n_1n_2A_1A_2$, and so $(m_1/n_1) \cdot (m_2)/(n_2) \in L(A_1A_2 : U_1U_2)$. Now we show that $L(A_1 : U_1) \cdot L(A_2 : U_2)$ is cofinal in $L(A_1A_2 : U_1U_2)$. Suppose $q \in L(A_1A_2{:}U_1U_2)$. We can assume that $q = m/n$, with:

$$m\, U_1U_2 \prec n\, A_1A_2 - 2\, U_1U_2. \tag{1}$$

Select k_2 and k_1 such that:

$$nA_1 \prec k_2U_1 \qquad \& \qquad nA_2 \prec k_1U_2. \tag{2}$$

Select s_1, s_2, $i = 1, 2$, such that:

$$s_iU_i \prec k_iA_i \preceq (s_i + 1)U_i. \tag{3}$$

Then $s_i/k_i \in L(A_i{:}U_i)$. We will show that $(s_1/k_1)(s_2/k_2) > m/n$. Set

$$E_i := k_iA_i - s_iU_i.$$

Then from (3), we get:

$$E_i \preceq U_i. \tag{4}$$

By (2) and (4),

$$n\, A_1\, E_2 \prec k_2U_1\, U_2 \qquad \& \qquad n\, A_2\, E_1 \prec k_1U_2U_1.$$

Multiplying these inequalities by k_1 and k_2 respectively and adding, we get:

$$n\,(k_1\,A_1\,E_2 + k_2\,A_2\,E_1) \prec 2\,k_1 k_2\,U_1 U_2.$$

Subtracting both sides from $n\,k_1 A_1\,k_2 A_2$ gives:

$$n(k_1 A_1\,k_2 A_2 - k_1\,A_1\,E_2 - k_2\,A_2\,E_1) \succ n\,k_1 k_2\,A_1 A_2 - 2\,k_1 k_2\,U_1\,U_2.$$

If we add nE_1E_2 to the left side, we preserve the inequality. Then we can factor the left side and rearrange the right side:

$$n(k_1 A_1 - E_1)(k_2 A_2 - E_2) \succ k_1 k_2\,(n\,A_1 A_2 - 2\,U_1 U_2).$$

Using the definition of E_i on the left and (1) on the right, we get:

$$n\ s_1 U_1\ s_2 U_2 \succ k_1 k_2\ m\ U_1 U_2.$$

Thus,

$$\frac{s_1\,s_2}{k_1\,k_2} > \frac{m}{n}. \qquad /////$$

Corollary. *Suppose* $A, W, U \in \mathcal{A}$. *Then*

$$L(A\!:\!W) \cdot L(W\!:\!U) = L(A\!:\!U).$$

In Hölder's theorem, the map from $\mathcal{A}$ to $\mathbb{R}$ is not unique. Each choice of U gives a different map

$$A \mapsto \sup L(A\!:\!U).$$

The corollary shows that the maps $A \mapsto \sup L(A:U)$ and $A \mapsto \sup L(A:W)$ differ from one another by a scalar factor.

8. Conclusions

We summarize the most important observations.

- The most basic objects in Euclid's theory of ratio and proportion are the so-called magnitudes. Segments, planar regions, volumes and angles are examples of magnitudes. If any two magnitudes of the same kind are compared with respect to size, they will either be found to be of the same size, or one will be bigger. Magnitudes of the same kind may also be added to one another—often in numerous different ways. Yet no matter how magnitudes are added, the size of the result depends only on the sizes of the ingredients. A smaller magnitude may be removed from a larger one of the same kind, also with no ambiguity regarding the size of the result. Adding or subtracting the same magnitude to or from two others preserve the size relation; of the two that are increased or diminished by the same magnitude, the larger remains larger.

- In speaking of the size of a magnitude, Euclid is not referring to a numerical measurement, but only to the ordering of the magnitudes of a given kind by size. Today, we measure objects to find a number that indicates size with respect to a standard, but no such measurement is presupposed by Euclid's theory.
- Given two magnitudes of the same kind, the ratio between them is a more precise indicator of relative size than the mere order (bigger/smaller). We may think of the ratio as the array of all comparisons of all possible multiples of the first magnitude with all possible multiples of the second. Thus, the ratio of a to b is characterized by the set of all pairs (m, n) such that $m\,b$ is less than $n\,a$ in size.
- In forming a ratio, the terms must be like magnitudes because direct comparisons of magnitude to magnitude are necessary. But once a ratio between objects of one kind it made, it may be compared to a ratio between objects of another kind. The ability to compare ratios of one kind of magnitude with ratios of another kind is one of the most powerful and important aspects of Euclid's theory.
- Euclid's theory applies to kinds of magnitude that satisfy the archimedean axiom: given any two different magnitudes, some multiple of the smaller exceeds the larger. In this setting, Euclid's theory entails that that

$$\begin{aligned}(A{:}B) \preceq_r (A'{:}B') \quad &\Leftrightarrow \quad \{\, m/n \mid m\,B \prec n\,A \,\} \subseteq \{\, m/n \mid m\,B' \prec n\,A' \,\} \\ &\Leftrightarrow \quad \sup\{\, m/n \mid m\,B \prec n\,A \,\} \leq \sup\{\, m/n \mid m\,B' \prec n\,A' \,\}.\end{aligned}$$

- Hölder demonstrated the following. Let $\mathcal{A}$ be an archimedean kind of magnitude. Fix a magnitude U in $\mathcal{A}$, and define a map $meas(\_\_, U) : \mathcal{A} \to \mathbb{R}_{\geq 0}$ by

$$meas(A, U) := \sup\{\, m/n \mid m\,U \prec n\,A \,\}.$$

  This map preserves the order and addition of magnitudes. The number $meas(A, U)$ may be thought of as "the measure of A with respect to U." Moreover, if W is another magnitude in $\mathcal{A}$, then for all A:

$$meas(A, W) = meas(A, U) \cdot meas(U, W).$$

- All the propositions that Euclid proved about ratios have analogues in the arithmetic of real numbers. We make the translation by selecting a standard unit magnitude U in each kind of magnitudes. In place of "the ratio of A to B", we think of "the number a/b", where a is the measure of A with respect to U and b is measure of B with respect to U.

9. A final comment.

Modern treatments of ratio and proportion are arithmetic/algebraic in form. The student works with equations and numbers. The numbers have a link to things. But this link is not "in the arithmetic", and often not on paper during the work. Experience suggests that it is often not in mind, either.

Consider, for example, the following problem:

> If 48 ounces of orange juice costs \$3.45, what should a quart (32 ounces) cost?

The arithmetic that gives the answer might start with the equation:

$$\frac{32}{48} = \frac{x}{3.45}.$$

The equation is significant because the 48 is a number of ounces of orange juice with known value and the 32 is also a number of ounces of the same thing, but of unknown value. And *the ratio* of 32 ounces of orange juice *to* 48 ounces of orange juice is the same as *the ratio* of the unknown cost *to* \$3.45. But the equation does not say this. It contains no hint of the origin of the numbers or their relationships to things. The equation, by itself, is just a statement about x.

The mathematics of Euclid is not a mathematics of numbers, but a mathematics of things. The symbols, relationships and manipulations have physical or geometric objects as their referents. You cannot work on this mathematics without knowing the objects that you are working with. A challenge for educators is to find a way of bringing this level of awareness of meaning into the modern, arithmetic/algebraic treatment in an appropriate way.