MINUS TIMES MINUS

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1. An Overview of the Different Approaches

Why is the product of two negative numbers positive? There are several different approaches for understanding the question.

- **Here Are The Integer Axioms. Now All Follows**. This is in some sense a "top-down" approach, and we take this approach in the separate piece "Rule of Signs in the Arithmetic of Integers". We start upfront with the assumption that the integers satisfy several axioms, and we deduce the sign of the product of two negatives from the axioms. This has the potential strengths of being rigorous and generalizable. More subtly, this rhetorical approach is standard in the practice of research mathematics. This does side-step the issue of why we should want those axioms, or whether one might define negative numbers differently.
- **It's the Best Way to Extend the Natural Numbers**. This is in some sense a "bottom-up" approach, which is developed below in "Adjoining Opposites to the Natural Numbers". We begin with the whole numbers with properties we know and love, and we contemplate adding in the negative integers as formal objects defined as additive inverses, but with no other assumed properties. A minimal number of properties of whole numbers are found to be non-negotiable (such as the distributive law), so we declare that the properties must extend to the new negative integers. From these extended properties, one deduces facts about these negative numbers. This has the potential strengths of being rigorous while addressing WHY we chose the axioms we did and whether there were sensible alternatives. This does side-step the issue of why the *whole numbers* have the properties they do and relies on people's greater comfort with natural numbers than negative numbers.
- **Construct the Integers Rigorously**. One can develop the theory of integers starting with only set theory, constructing the natural numbers and then constructing the integers. The advantage is that one might finally have the sense that one knows rigorously *exactly* what the integers are. The disadvantage is that there are a number of uninspiring technical details to check. The entry "A Rigorous Development of the Natural Numbers and Integers" attempts to give an outline of such a development while skipping the more boring parts.
- **It Makes Sense If You Follow a Metaphor**. This approach is developed in "The Rule of Signs in Contexts". We begin with a specific model (metaphor) of the integers, along with the operations of addition,

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subtraction and multiplication. The model needs to be accessible to some real-world intuition, even if it is not realistic. We convince ourselves that the metaphor is compatible with the properties of the integers. Then we look at the case of multiplying two negative numbers and using our understanding of the model, we see that the product of two negative numbers indeed ought to be a positive number. This is not a mathematically rigorous approach, but can often help people make intuitive sense of a perhaps strange idea.

- **Argument By Pattern**. One demonstrates that the natural numbers have certain arithmetic patterns and so the negative numbers ought to continue the patterns. This approach has the potential strengths of being convincing and accessible; the potential weakness is that inductive (as opposed to deductive) reasoning of this kind is not in itself mathematically rigorous. On the other hand, this kind of inductive reasoning can sometimes be made rigorous and deductive, as the patterns usually reflect an essential rather than a superficial aspect of our number systems. One such example is worked out in "A Rigorous Version of the Argument By Pattern".
- **Advanced Math Makes It Simple**. It is sometimes interesting and insightful to use some advanced mathematics to reflect on more basic mathematics. For instance, integer arithmetic is a special case of complex number arithmetic, and multiplication by a negative number is a particularly lovely special case. This approach is explored in "Multiplication as a Complex Plane Rotation" . "Minus Times Minus in More General Settings" reflects on "minus times minus" in the complex numbers and in matrix algebras.

2. Adjoining Opposites to the Natural Numbers

Take the natural numbers as the set we know and love equipped with addition and multiplication (and for simplicity, we count 0 as a natural number). For each natural number A let us define a new mysterious opposite number, $-A$, with the property $A+(-A) = 0 = (-A)+A$, i.e. $-A$ is an additive inverse of A. The union of the naturals and all their opposites we'll now call the integers. The nonzero natural numbers are called positive integers, and their opposites, negative integers.

We can already make one observation: The same defining property tells you that **each opposite $-A$ also has A as an additive inverse**, so for a general integer X, we'll write its additive inverse as $-X$ without guilt.

We would like to extend addition and multiplication to these new opposites in a way that extends essential properties of natural number arithmetic. We ask that they obey the distributive law, and that addition and multiplication are associative and commutative (as an aside remark, we do not really need commutativity to establish the rule of signs. However, when extending the natural numbers to the integers, we certainly *want* the operations to remain commutative). We also would like 0 to be an additive identity and 1 to be a multiplicative identity. It can be checked that the integers can be assigned these properties in a way that doesn't lead to contradictions. You can now deduce a lot about the integers from these basic properties.

Identities are unique. We automatically get the uniqueness of an identity by observing that if there were two identities $0, 0'$, then $0 = 0 + 0' = 0'$, with a similar argument for the uniqueness of 1.

Additive inverses are unique. That is, suppose A has both A' and A'' as two-sided additive inverses. Then

$$A' + A + A''$$

equals both A'' (if you do the left addition first) and A' (if you do the right addition first). Thus, A' and A'' were really the same object all along, so every element has a unique additive inverse.

The opposite of a number's opposite is the original number. We have already observed that A is an additive inverse of $-A$. By definition, $-(-A)$ is also an additive inverse of $-A$. By uniqueness, $A = -(-A)$.

Zero times anything is zero. For any integer B, $(0)(B) = 0$, because of the following observation. First, we agree that $(0)(B) + -(0)(B) = 0$. Also $0 + 0 = 0$ (since 0 is an additive identity), so we know the left-hand side equals $(0+0)B + -(0)(B) = 0(B) + 0(B) + -(0)(B) = 0(B)$, which must also equal the right side, 0.

We are now ready to see that the product of two negative integers is positive. We will take two integers A, B and show that the product of $-A$ and $-B$ is AB, which establishes the main result as a corollary.

First we'll show that $(-A)(-B)$ and $(A)(-B)$ are opposites.

$$\begin{aligned} (-A)(-B) + (A)(-B) &= (-A + A)(-B) && \text{distributive property} \\ &= (0)(-B) && \text{definition of } -A \\ &= 0 && \text{zero times anything is zero} \end{aligned}$$

Second, we'll show that $(A)(-B)$ and AB are opposites.

$$\begin{aligned} (A)(-B) + AB &= (A)(-B + B) && \text{distributive property} \\ &= (A)(0) && \text{definition of } -B \\ &= 0 && \text{zero times anything is zero} \end{aligned}$$

Since $(A)(-B)$ has a unique opposite, $(-A)(-B)$ equals AB.

Thus if you define the negative integers as additive inverses of (nonzero) natural numbers and want to keep a sensible arithmetic (i.e. want to preserve associativity, commutativity, distributivity, and keep 0 and 1 as the additive and multiplicative identities) then AB must equal $(-A)(-B)$, and in the special case where $-A$ and $-B$ are negative, their product is the product of two (nonzero) natural numbers, which is positive.

3. A Rigorous Development of the Natural Numbers and Integers

It is mathematically fulfilling to be able to construct the natural numbers in a rigorous way, though it is slightly tedious to check all the details. Here we outline a rigorous development beginning with only set theory, and using the Peano Axioms to construct the natural numbers and their arithmetic. We then use the Grothendieck Construction to construct the integers from the natural numbers.

3.1. Defining the Natural Numbers Using Set Theory.

A standard construction defines the natural numbers using the Peano Axioms (following Usiskin, et al, p189):

The natural number system is a set N, an element 0 of N and a "successor" function $s : N \to N$ with the properties

- 0 is not $s(n)$ for any $n \in N$. That is, 0 is not in the range of s.
- For any numbers $m, n \in N$, if $s(m) = s(n)$, then $m = n$. That is, s is a one-to-one function.
- If M is a subset of N which contains 0, and if $s(n) \in M$ for every $n \in M$, then $M = N$.

From this modest set of axioms, one can deduce that these objects can be named with the natural number names we expect, and that addition and multiplication can be defined with the results we expect. One elegant feature is that we can prove that addition and multiplication are commutative, associative, and have the distributive property, without resorting to adding any new axioms. Furthermore, the principle of mathematical induction is baked right into the original set of axioms. (Landau, *Foundations of Analysis* treats these steps in great detail.)

3.2. **Defining the Integers from the Natural Numbers.** Suppose we have constructed the natural numbers to our satisfaction. We gave a rigorous treatment in "Adjoining Opposites" by adjoining opposites as formal objects. Here we will use what is called the Grothendieck Construction. This is a very general construction which actually applies to any structure with a commutative, associative binary operation and an identity element and embeds it in a larger structure that extends the operation so that every element now has an inverse (i.e. embeds it in an abelian group). The set of natural numbers fits these preconditions, and we'll discuss this specific case, but this works for many other sets.

One defines new numbers of the form "$a-b$" where a and b are natural numbers. There is an important ambiguity to confront here because $3-2$ ought to be the same as $7-6$. Thus, we declare "$a-b$" to be equivalent to "$c-d$" if $a+d = b+c$. In fact, this is a genuine equivalence relation, so each integer is really an equivalence class of expressions "$a - b$". The natural numbers are embedded in the integers as the classes "$a - 0$", and negative numbers are the classes "$0 - a$", where a is a nonzero natural number. Addition is what you'd expect: $(a-b)+(c-d) = (a+c-(b+d))$.

Once the integers are constructed with an addition, it's natural to want to extend multiplication. One typical approach is to define multiplication on pairs of integers in the "natural" way: $(a-b)(c-d) = (ac+bd-(bc+ad))$. One does need to check that these operations are well-defined in that they do not depend on the choice of representative for the equivalence class. Details are available in your local abstract algebra text.

Given this definition, the product of two negative numbers $(0-b)(0-d)$, where b, d are (nonzero) natural numbers, is $(0 + bd - (0 + 0)) = (bd - 0)$, which is the natural number bd. Thus, the product of two negative numbers is positive.

4. A Rigorous Version of the Argument By Pattern

There is a common argument that

$$-1 \times 3 = -3$$
$$-1 \times 2 = -2$$
$$-1 \times 1 = -1$$
$$-1 \times 0 = 0$$

so it should be true that

$$-1 \times -1 = 1$$

This kind of inductive reasoning can be very persuasive to people, but raises red flags for mathematicians. Certainly there are any number of false conjectures which appeal to the fact that the first few numbers in a sequence have a property, so this property should extend to the rest. (One famous one was Fermat's conjecture that the Fermat numbers $F_n = 2^{2^n} + 1$ are all prime. Indeed, F_0, F_1, F_2, F_3, F_4 are all prime, but F_5 was proved composite by Euler.) In this case, the pattern is structured enough that we can turn this into a rigorous argument.

4.1. **Special Case: $-1 \times -1 = 1$.** The meat of the informal argument is each product is one more than the last one. More precisely, the argument is that

$$(-1)(N) = (-1)(N + 1) + 1$$

is an identity and specifically for $N = -1$ we get that $(-1)(-1) = 1$.

We will use only the assumptions that (a) the integers have the distributive property, (b) 1 ought to remain a multiplicative identity, and (c) -1 is the additive inverse of 1. First,

$$(-1)(N + 1) + 1 = (-1)(N) + (-1)(1) + 1$$

by the distributive property.

Now we will be finished if we can show the last two terms sum to zero. Indeed,

$$(-1)(1) + 1 = -1 + 1 = 0$$

by the property that 1 is a multiplicative identity, and that -1 is the additive inverse of 1. Thus the pattern does indeed hold as a consequence of the three reasonable assumptions.

When you plug in $N = -1$ to this identity, you get

$$(-1)(-1) = (-1)(-1 + 1) + 1.$$

The right hand side equals

$$(-1)(0) + 1 = 1,$$

because in any ring, products with zero are zero (see "Adjoining Opposites"). Thus, $(-1)(-1) = 1$.

4.2. **General Case: $-a \times -b = ab$.** For the general case, one can follow several arguments deducing it from the special case of (-1)(-1) established above. Instead, we now make a pattern argument similar to the previous one, using numbers a, b instead of -1.

$$\begin{aligned} -a \times 1 &= -a \\ -a \times 0 &= 0 \\ -a \times -1 &= a \\ &\vdots \\ -a \times -(b-1) &= a(b-1) \end{aligned}$$

so it should be true that

$$-a \times -b = (a)(b).$$

This pattern argument can be made rigorous in exactly the same way as in the last section. You establish the analogous identity as follows for any integers M, N:

$$(-M)(N+1)+M = (-M)(N)+(-M)(1)+M = (-M)(N)+-M+M = (-M)(N).$$

The justifications of each equality are exactly the same as in the special case calculation.

Then starting from the base case of $M = a$ and $N = -b$, you get

$$(-a)(-b+1) + a = (-a)(-b).$$

Now we can apply the identity with $N = -b + 1$. Repeat the identity b times to expand it all the way to

$$(-a)(0) + a + \ldots + a = (-a)(-b),$$

where there are b terms of a on the left side. The left hand side simplifies to $(a)(b)$.

5. Multiplication as a Complex Plane Rotation

The complex plane gives a beautiful way to think about multiplying negative numbers, and in particular multiplying by -1.

First let us recall that a complex number can be written in polar form, $re^{i\theta}$, and this number is mapped to a point in the complex plane which is the endpoint of a line segment starting at the origin of length r and forming an angle with the positive x-axis equal to θ radians.

Multiplication of two complex numbers results in a complex number whose magnitude is the product of the original magnitudes, and whose angle is the sum of the original angles. For our key case, $-1 = e^{i(\pi)}$. Thus, $(-1)(-1) = e^{i(2\pi)}$. This last point is one unit at a 2π angle with the positive x-axis, i.e. 1.

A pleasant interpretation of multiplying by a complex number $re^{i\theta}$ is as a scaling of the original quantity by r and a rotation by θ radians counterclockwise around the origin. Now consider the product of $-a$ and $-b$, where a and b are natural numbers. These can be written as $ae^{i\pi}$ and $be^{i\pi}$. So $-a$ sits on the negative x-axis, a units to the left. Multiplying by $-b$ results in scaling the original length of a to a new length of ab, and rotating it π radians, i.e. halfway around to the positive x-axis. Thus, $(-a)(-b) = ab$.

A more sophisticated take on this interpretation is to consider the complex numbers as a set of transformations of the complex plane. Each complex number a is an operator that maps $a : x \mapsto ax$. The operator (-1) acts on the complex plane by rotating every point halfway around the origin. Without going into technical details (which can be found in a good operator theory book), C acts on itself as a complex algebra of multiplication operators. In this setting "multiplication" of two operators is defined as their composition, but ends up being equivalent to their complex product. Thus, $(-1)(-1)$ is an operator that rotates every point halfway around, then rotates those rotated points halfway around again, which naturally returns them to their original location. Hence $(-1)(-1)$ is the identity operation, which is the multiplication operator corresponding to 1.

One cannot typically use the theory of complex numbers as a *logical* foundation for the integers (since typical developments of the complex numbers rely on properties of the integers), but the advanced perspective gives "minus times minus" a nice context.

6. Minus Times Minus in More General Settings

The rule "minus times minus is plus" is learned before students know about complex numbers, and often it is described a little less loosely as "a negative times a negative is positive", or "the product of two negative numbers is positive". Clearly the terminology refers to numbers on the number line. However, a more general statement can be made, which also applies to complex numbers. For this, we need to replace the concept of negative with that of the additive inverse, or opposite, of a number. The more general rule is: the product of the opposites of two number equals the product of the two numbers..

On the number line, the opposite of a positive number is negative and vice-versa. Geometrically, a real number and its opposite are equidistant and on opposite sides from zero on the number line. More generally, in the complex plane, a number and its opposite are the endpoints of a line segment that has zero as its midpoint. We adopt the usual notation for the opposite of a, $-a$.

With this notation, the rule becomes $(-a)(-b) = ab$. This is true for any two complex numbers (in particular, they can be real). In fact, the rule applies to other algebraic structures as well. One such structure studied in high school is that of $n \times n$ (square) matrices (typically, 2×2 or 3×3 with real entries, which appear in connection to solving systems of linear equations). Interestingly, multiplication of matrices is not commutative, yet the rule still holds true because the arithmetic properties of these matrix structures is sufficiently similar to that of the complex numbers.

One can prove the rule for complex numbers by establishing that $-a = (-1)a$ and $(-1)(-1) = 1$. As a result, $(-a)(-b) = (-1)a(-1)b = (-1)(-1)ab = ab$. Interestingly, you can even use this approach for matrices! This argument seems to require that multiplication be commutative, and hence it seems not to apply to matrix algebras (which do not commute except for the 1×1 case). However, this approach really only requires that multiplication with 1 and -1 be commutative, and this *is* true for matrices (where 1 is the multiplicative identity matrix I).