Graphs of functions and equations

Author: *Stanley* General topic: *G. Functions and equations* Version: *2* Date: *September 17, 2008*

This note gives an approach to showing how functions and equations are related to each other, and how they are different. In terms of this approach a reason is suggested for why the two terms are often used interchangeably in imprecise ways in high school mathematics: the graph of a function of one variable is identical to the set of solutions of an equation in two unknowns.

Briefly, we take as primary the idea of a function as a mapping x --> f(x) from a domain set to a range set, and say that *an equation based on such a function* is of the form f(x) = k for any element k of the range. We view an equation as a way of stating a condition on the domain set of the function on which it is based, a condition met by the solutions of the equation. Often, k has the value 0, which leads to the "normal form" f(x) = 0 of an equation. A simple example is the function $x \rightarrow x^2 - 1$ and the related equation $x^2 - 1 = 0$.

Evaluating a function means finding the unique output value that comes from a given input value in the function's domain. For example, the function x --> x^2 - 1 is evaluated at x = 7 by calculating $(7)^2 - 1 = 48$.

Solving an equation based on a function means finding the value(s) of the input that the function maps to a given output. It is in this sense that an equation states a condition on the domain set of a function. For example, the solutions of the equation $x^2 - 1 = 99$ are x = 10 and x = -10 because these numbers (and only these) are mapped to the given output 99 by the function x --> x^2 - 1, and so meet the condition stated by the equation.

In short, evaluating a function matches an output to a given input, while solving an equation matches an input (or inputs) to a given output of a function. We never speak of solving a function or evaluating an equation.

Our principal observation is that **graphically**, the same graph can be interpreted one way in terms of evaluations of functions and another way in terms of solutions of an equation. This can be a source of conceptual and terminological confusion.

To sort out the issues, we need to extend the ideas above to functions of 2 variables and to equations of 2 unknowns based on them. In fact, consider the table below, which gives a systematic framework for thinking about functions of any number *n* of variables, equations of *n* unknowns, and their relationship.

	name of mathematical object	symbolic representation	what the object is
1a	a function of 1 variable	$x \rightarrow f(x)$	a mapping $\mathbf{R} \Rightarrow \mathbf{R}$
1b	a graph of this function	**$\{ (x, y) \mid y = f(x) \}$**	**a curve in the x-y plane**
1c	an equation in 1 unknown based on this function	$f(x) = 0$	a condition on numbers $x \in \mathbf{R}$
1d	solutions of this equation	$\{ x \mid f(x) = 0 \}$	a set of points on the x-axis
	functions and equations based on the expression f(x)		
2a	a function of 2 variables	$(x, y) \rightarrow F(x, y)$	a mapping $\mathbf{R}^2 \Rightarrow \mathbf{R}$
2b	a graph of this function	$\{ (x, y, z) \mid z = F(x, y) \}$	a surface in x-y-z space
2c	an equation in 2 unknowns based on this function	$F(x, y) = 0$	a condition on pairs $(x, y) \in \mathbf{R}^2$
2d	solutions of this equation	**$\{ (x, y) \mid F(x, y) = 0 \}$**	**a curve in the x-y plane**
	functions and equations based on the expression F(x, y)		
3a	a function of 3 variables	$(x, y, z) \rightarrow G(x, y, z)$	a mapping $\mathbf{R}^3 \Rightarrow \mathbf{R}$
3b	a graph of this function	$\{ (x, y, z, w) \mid w = G(x, y, z) \}$	a 3-d subset of x-y-z-w space
3c	an equation in 3 unknowns based on this function	$G(x, y, z) = 0$	a condition on triples $(x, y, z) \in R^3$
3d	solutions of this equation	$\{ (x, y, z) \mid G(x, y, z) = 0 \}$	a surface in x-y-z space
	functions and equations based on the expression G(x, y, z)		

The table shows that **the graph of a function of 1-variable** and the **solutions of an equation in 2 unknowns** are both the same sort of object, namely, **a curve in the x-y plane**. See the right-most column of row 1b and row 2d.

For example, consider the function of two variables $(x, y) \to F(x, y)$ defined by $F(x, y) = 4x - y - 8$. The equation $F(x, y) = 0$, which we can write explicitly as $4x - y - 8 = 0$, has as its solution the line in the x-y plane with intercepts (0, -8) and (2, 0). On the other hand, the function of one variable defined by $f(x) = 4x - 8$ has as its graph the very same line.

Central to this discussion is the fact that the x-y plane serves two different kinds of roles.

1. On the one hand, curves in the x-y plane are described algebraically by equations of the general form $F(x, y) = 0$, and their geometric properties are studied via these equations. This is the traditional subject of elementary analytic geometry.

2. On the other hand, the study of analytic properties of functions of one variable is enhanced by giving them a geometric representation; this is done through their graphs, which are curves in the x-y plane.

These two subjects are distinct. For example, the *slope* of a curve at a point is very much part of (2), but not of (1). At the same time, the *length* of a curve is part of (1), but not of (2). And even though the *area* enclosed by a curve or curves may be studied in either, it plays a very different role in each.

We note that functions and their graphs are receiving increased attention in school mathematics in recent years, while analytic geometry receives less emphasis than it once did. In the process of this evolution, the distinctness of the subjects seems to be lost. There is a tendency to regard expressions containing x and y and an = sign as mere strings of symbols to be operated on procedurally. In such an environment it is no wonder that such strings are referred to indifferently as *equations* or as *functions*.