3 Views of F(x, y) = 0: differing uses of the terms equation and function

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general topic: *G. Functions and equations*
version: *2*
date: *18 September 2008 (with corrections added 21jan09)*

Introduction

Symbolic forms with two variables x and y of the general form F(x, y) = 0 appear frequently in high school mathematics. Such a form can be looked at from three rather different points of view, corresponding to the fields of analysis, algebra, and analytic geometry. We take as an illustration the case F(x, y) = 4x - y - 12, and show the different forms and meanings the expression 4x - y - 12 typically has in each subject.

(1) In **analysis**, a representation $y = y_0 + kx$ is a formula for a ***function***.
y = 4x - 12 defines a linear function of one variable x.

(2) In **algebra**, a representation ax + by = c is an ***equation.***
4x - y = 12 is a linear equation in two unknown*s* x and y.

(3) In **analytic geometry**, a representation $\frac{x}{a}+\frac{y}{b}=1$ is the ***equation*** of a line.
$\frac{x}{3}+\frac{y}{-12}=1$ is the equation of a line with x-intercept 3 and y-intercept -12.

We give brief discussions that highlight the different roles functions and equations play at the high school level in each of the fields (1) to (3). We will continue to use 4x - y - 12 = 0 as an example, but our remarks will apply as well to more general types of equations and functions involving real numbers.

1. *A function of one real variable in analysis:*

In analysis it is common to use a symbolic form such as y = 4x - 12 to define a particular *function*. Mathematically what is being described is an analytic relationship between two co-varying quantities x and 4x - 12. A more logical notation is x ◊ 4x - 12, indicating that a function is a *mapping* of one set to another. Function notation in a *formula* such as y = 4x - 12 is also useful in that it gives a single symbol for the output. But calling y = 4x - 12 an *equation* runs the risk of confusing this with the equations of algebra, which play a rather different role.

What y = 4x - 12 and x ◊ 4x - 12 have in common is the use of an *expression* 4x - 12 in defining a function. In fact, one way to give a general mathematical definition of *function* is in terms of formulas of the general form y = f(x), where, f(x) is an *expression in x*. If we define functions in this way, everything depends on what is meant by *an expression in x*. Typically included are "algebraic" expressions, meaning any well-formed strings consisting of the variable *x* and numbers connected by arithmetic operations of addition, subtraction, multiplication, division, and exponentiation.

Such a definition is sufficient to include many functions studied in high school, but certainly not all. A major exception is logarithmic functions, which cannot be defined using (finite) expressions of this form. But they can be defined as inverses of exponential functions, and this is a mathematically important way of defining new classes of functions from given classes which can be discussed in high school.

Another exception is trigonometric functions and their inverses. These form a class for which there is no definition of the form y = f(x) for finite expressions f(x) in x defined algebraically.

To address this lack of algebraic formulas for such functions, a definition of *function* in terms of sets of ordered pairs is often given. This is a completely general definition, but for this very reason it does nothing to shed light on the particular functions that are useful to study in high school. Based on such an ordered pair definition there are procedures such as the "vertical line test" that can be carried out, but few theorems that can be proved.[1]

The question arises, then, of what is an appropriate approach to a general definition of the types of functions useful to study at the high school level.[2]

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[1] The main theorems are technical results connecting the ideas for functions of 1-1, onto, inverse, left inverse, and right inverse.

[2] Using power series, a wide variety of functions can be defined in terms of analytic expressions in x, and significant theorems can be proven about them. Although power series do not appear before calculus, one wonders if an earlier introduction, perhaps in the form of polynomial approximations, might not allow the study of functions in high school to have more coherence.

2. *An equation in 2 unknowns in algebra:*

In algebra it is common to view a symbolic form such as 4x - y = 12 as an ***equation*** *in two unknowns.* A *solution* of such an equation is a pair (x_0, y_0) of numbers that satisfies $4x_0 - y_0 = 12$. The set of all such solutions forms a line in the x-y plane.

Such an equation is often treated as one of a pair of two equations such as

(4a) 4x - y = 12 (4b) 2x - y = 4

Each equation places a condition on x and y, and what is sought is a pair (x_0, y_0) that satisfies each of the two equations. Such a pair solves them *simultaneously.* A single such equation is never sufficient to determine a definite value (x_0, y_0) for x and y, but two conditions often are sufficient.

It is convenient to discuss solutions to equations such as (4) in terms of their graphs as lines in the plane. There is a unique solution (x_0, y_0) if the lines are not parallel, no solutions if the lines are distinct and parallel, and an infinite number of solutions if the lines are identical.

Although linear functions in one variable and linear equations in two unknowns happen each to have a graph that is a line in the x-y plane, there is an enormous difference in the role they play in analysis and algebra. Here is an amusing way of emphasizing this difference.

Suppose we express the *equations* (4), which are in the general form ax + by = c, in the "equivalent" form (5) as formulas y = mx+ b for *functions,* as in Section (1):

(5a) y = 4x - 12 (5b) y = 2x - 4

Now (4a) and (5a) are, in fact, equivalent in the sense that they have the ***same graph***. And (4b) and (5b) also have the same graph. However, the *sum* of (4a) and (4b), viewed as equations, has a different graph from the *sum* of (5a) and (5b), viewed as functions.

Specifically, viewing (4a) and (4b) as *equations*, we can take their sum 2y - 5x + 6 = 0 and graph this new equation.[3] Next, viewing (5a) and (5b) as formulas for *functions* a(x) and b(x), we can take their sum a(x) + (b)(x), and graph this new function.

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[3] Taking sums and differences of equations in this way is common in procedures for solving equations. The important fact is that any linear combination of two equations E_1 and E_2 has a graph that passes through the intersection of the graphs of E_1 and E_2.

It will be seen that the graph of the equation sum is ***different*** from the graph of the function sum. This is very surprising to people who are used to using the terms "function" and "equation" interchangeably. It is not surprising if functions are viewed as mappings and equations as conditions.

To return to the general discussion of algebra, we note that there are general procedures for solving (or showing that there is no solution for) any two linear equations in two unknowns, such as (4). This is the beginning of the study of *linear algebra*. Some of the mathematics of that subject is often introduced in high school.

3. *An equation describing a geometric object in analytic geometry:*

From the perspective of *analytic geometry*, a relationship such as $\frac{x}{a}+\frac{y}{b}=1$ is an *equation.* It describes a geometric object, in this case a *line* in the x-y plane. It is customary to refer to a symbolic form such as $\frac{x}{a}+\frac{y}{b}=1$ as the *symmetric form* of the equation of a line.

Analytic geometry gives descriptions of *conic sections.* This class of geometric objects in the x-y plane includes all lines, circles, parabolas, hyperbolas, and ellipses, and nothing else. In fact, members of this class can be described in three very different ways:

- As a curve formed by the intersection of a plane and a cone.
- As the "locus" of a point in terms of its distance from a fixed point and another point or line.
- As the set of points satisfying a 2nd degree equation $F(x, y) = 0$.

The fact that these three approaches define the same class of curves is a significant result that can be developed at the high school level.

4. *Comparison of the three perspectives*

In high school the three perspectives of analysis, algebra, and analytic geometry are often blurred in treatments of functions and equations. We have already seen, at the end of §2, an example of the type of peril associated with this blurring.

We have also emphasized the different typical forms used in each subject. Consider

- "y = f(x) form" $y = mx + b$
- "symmetric form" $\frac{x}{a} + \frac{y}{b} = 1$
- "standard form" $Ax + By + C = 0$

Each is a way of describing what is graphically the same sort of object: a *line in the plane*. However, their different forms correspond to their different roles in the three fields.

The symmetric form is especially useful in analytic geometry, where it shows the parallel with equations of circles and ellipses. The standard form is common in algebra, and is also useful in analytic geometry when extended to include second degree terms.

Finally, the y = f(x) form, the only one not symmetric in x and y, is useful in stating formulas for functions, but is not particularly suited for algebra or analytic geometry.

Perhaps the individual character of these fields would be appreciated more at the high school level if something of the history of these fields was discussed. Analytic geometry is a field that existed in a mature form in the 17th century at a time before functions were used at all in mathematics. The field of analysis originated in applied settings at the beginning of the scientific revolution, and was later formalized in pure mathematics, with functions coming into their own in the 18th and 19th centuries. Linear algebra was an even later development.

5. *Two points of view*

We note that the treatment of functions, and equations in each of these three fields can be looked at from two points of view:

(A) ***Definitions and procedures***:
How are the symbolic forms that constitute expressions, functions, and equations correctly used, and what specific procedures can be carried out on them?

(B) ***Theorems and implications***:
What general theorems can be proved that apply to expressions, functions, and equations, and what are the important overall mathematical implications?

Study of any field must begin with (A), but ultimately a field is of mathematical interest only through (B).

Here are some examples of what we mean by (A) and (B) at the level of high school mathematics for the fields (1), (2), and (3).

(1a) An example of a procedure in analysis is evaluating an expression such as $4x - 12$ at $x = -3$. Another is finding the minimum value of an expression such as $x^2 - x - 1$.

(1b) An example of a theorem in analysis is that the graph of a "linear" function x ◊ kx + b is geometrically a line.

(2a) An example of a procedure in algebra is solving a quadratic equation.

(2b) An example of a theorem in algebra is that every cubic equation with real coefficients has a real root.

(3a) An example of a procedure in analytic geometry is finding the center and radius of a circle given its equation.

(3b) An example of a theorem in analytic geometry is that second degree algebraic equations $F(x, y) = 0$ describe curves that are sections of a cone.

Unless significant examples of ***Theorems and implications*** are in the forefront of the discussion of high school mathematics, it is difficult to reveal the true mathematical character of these fields at the high school level. One of the challenges in considering appropriate content for high school mathematics is how to have a meaningful balance between (A) ***Definitions and procedures*** and (B) ***Theorems and implications***. This is an issue that goes far beyond what can be discussed in the present note.